ON A CLASS OF QUINTIC DIOPHANTINE EQUATIONS IN TWO UNKNOWNS

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Inaugural Dissertation

BY

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PREFACE

In publishing this dissertation I wish to express my deep gratitude to my teacher in mathematics at Uppsala university, Professor TRYGVE NAGELL, who initiated me in the theory of numbers and in particular in the study of Diophantine equations. I am deeply indebted to Professor NAGELL for the valuable guidance he has given me both during seminars and privately, and for the encouraging interest he has shown in my studies.

I should also like to thank Professor THORALF SKOLEM, Oslo, for a stimulating correspondence on the subject of this paper.

Finally I wish to express my thanks to Fil. lic. LARS FJELLSTEDT for the exacting work of reading the proofs, and to Mr. W. K. PERCIVAL who has helped me with the translation.

Uppsala, August 1952.

PER HÄGGMARK.

INTRODUCTION

Let $f_n(u, v)$ be a binary irreducible form of degree $n \geqq 3$ with coefficients which are rational integers. According to a famous theorem proved by A. THUE [1][1], the Diophantine equation

$$f_n(u, v) = 1$$

has only a finite number of solutions in rational integers u and v.

To obtain more clearly defined results, it has been necessary to specialize the form f_n. A series of investigations of this kind have been made on among others the Diophantine equation

$$A u^n - B v^n = C. \tag{1}$$

THUE [2] treated this equation in cases where certain conditions were satisfied. The results of THUE are included in the following proposition proved by C. SIEGEL [1]:

a, b, c may be integers $c > 0$, $n \geqq 3$. *If*

$$(a b)^{2^{n-1}} \geqq c^{2n-2} \cdot 4 \left[n \prod_{p \mid n} p^{\frac{1}{p-1}} \right]^n,$$

then the inequality

$$| a x^n - b y^n | \leqq c$$

has at most one solution in positive integers x *and* y, $(x, y) = 1$.

SIEGEL obtains this result with the aid of theorems from the theory of hypergeometrical functions.

Without using the method of THUE and with arguments taken exclusively from the theory of numbers, T. NAGELL [1] and B. DELAUNAY [1][2] have treated (1) making $n = 3$. With a method anal-

[1] Figures in [ ] refer to the Bibliography at the end of this paper.

[2] For further literature references see NAGELL, L'analyse indéterminée de degré supérieur, Mém. Sci. math. Fasc. 39, 1929.

ogous to that used by NAGELL, W. LJUNGGREN [1, 2] has treated the cases $n=4$ and $n=6$.

The method of NAGELL is based on the theory of units in cubic fields and further mainly on arguments of divisibility. In the above-mentioned treatises of NAGELL and LJUNGGREN an important fact was that in the fields where the operations were made, the field was either primitive and had only one fundamental unit or it was imprimitive but had a subfield with only one fundamental unit.

TH. SKOLEM, in a series of papers [1, 2, 3, 4][1], has developed a method, the so-called p-adic method, by which it is often possible to treat Diophantine equations also in cases where the operations have to be made, for instance, in a primitive field with more than one fundamental unit. The p-adic method is an extension and generalization of NAGELL's method. It is in the first place a method for the treatment of certain exponential equations. Its applicability to Diophantine equations of a non-exponential type is due to the fact that, by virtue of the theory of units, great classes of such equations are equivalent to certain exponential equations.

In the present paper we will treat the case $n=5$ and use the p-adic method to find a low upper limit to the number of integral solutions of the quintic Diophantine equation

$$u^5 + M v^5 = 1. \tag{2}$$

SKOLEM [2] has treated (2) for $M=2$, 4, 8, and 16 and demonstrated that (2) has only the trivial solution $u=1$, $v=0$ for $M=4$, 8 and 16, and further, when $M=2$ the only non-trivial solution $u=-1$, $v=1$. SKOLEM obtained this as a special result while trying to find a low upper limit to the number of integral solutions of certain ternary forms. The proofs were based on studies of the units of the ring $\boldsymbol{R}(\sqrt[5]{2})$ where a system of fundamental units was known.

We shall prove the following theorem:

The Diophantine equation (2) has at most two non-trivial integral solutions.

---

[1] See also SKOLEM, Diophantische Gleichungen, Ergebnisse der Math. und ihrer Grenzgebiete, Bd V, Heft 4, 1938 (Reprinted, New York 1950).

In order to prove this we shall study the units of the ring $\boldsymbol{R}(\sqrt[5]{M})$. However, we shall not start from a system of fundamental units of $\boldsymbol{R}(\sqrt[5]{M})$ as was done in the above investigation by SKOLEM. Instead we shall consider a system of independent units of the ring. In fact, in Chapter II we shall prove the theorem:

If (2) has two non-trivial solutions, then the corresponding units of $\boldsymbol{R}(\sqrt[5]{M})$ *form a system of independent units of* $\boldsymbol{R}(\sqrt[5]{M})$.

We shall then use this result in Chapter III to prove that (2) cannot have three integral solutions.

In Chapter I an account is given of those theorems, deduced from the theory of p-adic numbers, which will be used later on.

It follows from THUE-SIEGEL's theorem that (2) has at most one non-trivial solution if $M > 2 \cdot 5^4 \cdot \sqrt[6]{20}$. From a result by DIRICHLET [1] and LEBESGUE [1] concerning the equation $x^5 + y^5 = A z^5$ it follows that (2) has no integral solution u and v, $|uv| > 1$, if M is not divisible by a prime factor of the form $5m+1$ and if

$$M \equiv \pm 2,\ \pm 3,\ \pm 4,\ \pm 5,\ \pm 6,\ \pm 8,\ \pm 9,\ \pm 10,\ \pm 11,\ \pm 12 \pmod{25}.$$

SIEGEL's method of treating equation (1) seems artificial and cannot be generalized to forms of more than two variables. In fact, it would seem impossible to use his argument where the form cannot be written as a binomial expression equal to a constant. To arrive at his conclusions SIEGEL makes use of results from the theory of functions.

The p-adic method is the natural method of treating comprehensive classes of Diophantine equations. It belongs entirely to the theory of numbers. Moreover, it is naturally desirable to avoid arguments taken from other branches of mathematics.

Using p-adic developments enables us among other things to study to what extent the following equations are simultaneously solvable and unsolvable

$$u^5 + M v^5 = 1,\ u^5 + M^2 v^5 = 1,\ u^5 + M^3 v^5 = 1,\ u^5 + M^4 v^5 = 1.$$

Moreover (Cp. SKOLEM [2, 4 Satz 9]) it is possible to find a low upper limit to the number of integral solutions of, for example, the equation (N denotes the norm)

$$N(u + v\sqrt[5]{M} + w\sqrt[5]{M^2}) = 1.$$

In this paper we have merely taken the first step towards a more detailed study of quintic Diophantine equations. Since the proofs tend to be lengthy we have had to confine ourselves to studying equation (2).

It is my intention, however, to return to these other questions at some later time.

CHAPTER I

Some theorems concerning systems of p-adic equations

1. Th. Skolem [1, 3] has proved the following

Theorem 1. *Let p be a rational prime, and let*

$$f_0(x),\ f_1(x),\ f_2(x),\ \ldots$$

be polynomials with coefficients which are rational integers modulo p. If $f_0(x)$ is of degree n and contains at least one coefficient not divisible by p, there are at most n rational integers x satisfying the infinite series of congruence relations

$$\sum_{i=0}^{\nu} p^i f_i(x) \equiv 0 \pmod{p^{\nu+1}}, \quad \nu = 0, 1, 2, \ldots \tag{1}$$

As a matter of fact, this theorem is identical with the following theorem concerning p-adic power series: If $P(x) = \sum_{n=0}^{\infty} a_n x^n$ is a p-adic power series converging for all p-adic integers x, then the equation $P(x) = 0$ has only a finite number of solutions in p-adic integers if $a_i \not\equiv 0\ (p)$ for at least one i.

Skolem proves Theorem 1 by demonstrating that relations (1) are equivalent to an algebraic equation in x of degree n with integral p-adic coefficients. The theorem follows from the fact that such an equation has at most n solutions in p-adic integers. (Cp. Hensel [1].)

Skolem [1] has further proved

Theorem 2. *Let p be a rational prime, and let*

$$F_1(x, y) = \sum_{i=0}^{\infty} p^i f_{i1}(x, y), \quad F_2(x, y) = \sum_{i=0}^{\infty} p^i f_{i2}(x, y),$$

where f_{i1} and f_{i2} are polynomials with coefficients which are rational integers modulo p. If we have a k such that the congruence relations[1]

$$F_1^{(k)} \equiv 0, \quad F_2^{(k)} \equiv 0, \quad \frac{\partial F_1^{(k)}}{\partial x}\frac{\partial F_2^{(k)}}{\partial y} - \frac{\partial F_1^{(k)}}{\partial y}\frac{\partial F_2^{(k)}}{\partial x} \equiv 0 \pmod{p^{k+1}}$$

are not solvable simultaneously, then the system of p-adic equations

$$F_1 = 0, \quad F_2 = 0 \,(p)$$

has only a finite number of solutions in rational integers x and y.

2. We shall now prove the corresponding theorem in the case of three variables x, y and z. The method of demonstration is SKOLEM's.

Let p be a rational prime and let us suppose that we have a system of p-adic equations

$$F_1(x, y, z) = 0, \quad F_2(x, y, z) = 0, \quad F_3(x, y, z) = 0 \,(p), \tag{2}$$

where

$$F_r(x, y, z) = \sum_{i=0}^{\infty} p^i f_{ir}(x, y, z)$$

for $r = 1, 2, 3$, and where f_{ir} are polynomials with coefficients which are rational integers modulo p. The relations (2) imply

$$F_r^{(\nu)}(x, y, z) = \sum_{i=0}^{\nu} p^i f_{ir}(x, y, z) \equiv 0 \pmod{p^{\nu+1}}$$

for $r = 1, 2, 3$ and $\nu = 0, 1, 2, \ldots$

Let us further suppose that there is a k such that

$$\frac{\partial (F_1^{(k)}, F_2^{(k)}, F_3^{(k)})}{\partial (x, y, z)} \not\equiv 0 \pmod{p^{k+1}}. \tag{3}$$

Let a, b, c be a particular solution of (2) and x, y, z an arbitrary solution such that

$$x - a = x' \equiv 0, \; y - b = y' \equiv 0, \; z - c = z' \equiv 0 \pmod{p^{k+1}}. \tag{4}$$

[1] $F_j^{(k)} = \sum_{i=0}^{k} p^i f_{ij}(x, y), \; (j = 1, 2).$

TAYLOR's formula for polynomials in three variables gives

$$(5)\quad \begin{cases} F_1^{(\nu)} \equiv x'\left(\dfrac{\partial F_1^{(\nu)}}{\partial x}\right) + y'\left(\dfrac{\partial F_1^{(\nu)}}{\partial y}\right) + z'\left(\dfrac{\partial F_1^{(\nu)}}{\partial z}\right) + \cdots \equiv 0 \pmod{p^{\nu+1}}, \\ F_2^{(\nu)} \equiv x'\left(\dfrac{\partial F_2^{(\nu)}}{\partial x}\right) + y'\left(\dfrac{\partial F_2^{(\nu)}}{\partial y}\right) + z'\left(\dfrac{\partial F_2^{(\nu)}}{\partial z}\right) \equiv 0 \pmod{p^{\nu+1}}, \\ F_3^{(\nu)} \equiv x'\left(\dfrac{\partial F_3^{(\nu)}}{\partial x}\right) + y'\left(\dfrac{\partial F_3^{(\nu)}}{\partial y}\right) + z'\left(\dfrac{\partial F_3^{(\nu)}}{\partial z}\right) + \cdots \equiv 0 \pmod{p^{\nu+1}}, \end{cases}$$

where $x=a$, $y=b$, $z=c$, are to be put in the derivatives of (5). The following terms of $F_r^{(\nu)}$ are of a degree $\geqq 2$ in x', y', and z'. The congruence relations (5) may also be written

$$(6)\quad \begin{cases} A_0 + A_1 p^{k+1} + A_2 p^{k+2} + \cdots \equiv 0 \pmod{p^{\nu+1}}, \\ B_0 + B_1 p^{k+1} + B_2 p^{k+2} + \cdots \equiv 0 \pmod{p^{\nu+1}}, \\ C_0 + C_1 p^{k+1} + C_2 p^{k+2} + \cdots \equiv 0 \pmod{p^{\nu+1}}, \end{cases}$$

where

$$A_0 = x'\frac{\partial F_1^{(k)}}{\partial x} + y'\frac{\partial F_1^{(k)}}{\partial y} + z'\frac{\partial F_1^{(k)}}{\partial z} + \cdots,$$

$$A_j = x'\frac{\partial f_{k+j,1}}{\partial x} + y'\frac{\partial f_{k+j,1}}{\partial y} + z'\frac{\partial f_{k+j,1}}{\partial z} + \cdots, \quad (j \geqq 1).$$

$$B_0 = x'\frac{\partial F_2^{(k)}}{\partial x} + y'\frac{\partial F_2^{(k)}}{\partial y} + z'\frac{\partial F_2^{(k)}}{\partial z} + \cdots,$$

$$B_j = x'\frac{\partial f_{k+j,2}}{\partial x} + y'\frac{\partial f_{k+j,2}}{\partial y} + z'\frac{\partial f_{k+j,2}}{\partial z} + \cdots, \quad (j \geqq 1).$$

$$C_0 = x'\frac{\partial F_3^{(k)}}{\partial x} + y'\frac{\partial F_3^{(k)}}{\partial y} + z'\frac{\partial F_3^{(k)}}{\partial z} + \cdots,$$

$$C_j = x'\frac{\partial f_{k+j,3}}{\partial x} + y'\frac{\partial f_{k+j,3}}{\partial y} + z'\frac{\partial f_{k+j,3}}{\partial z} + \cdots, \quad (j \geqq 1).$$

In all the derivatives we make $x=a$, $y=b$, $z=c$. Suppose that

$$A_0 = p^\mu \bar{A}_0, \; B_0 = p^\mu \bar{B}_0, \; C_0 = p^\mu \bar{C}_0, \; (\bar{A}_0, \bar{B}_0, \bar{C}_0, p) = 1.$$

It follows from (6) that $\mu \geqq k+1$. We form the expressions

$$
\left|\begin{array}{l}
A_0 \dfrac{\partial (F_2^{(k)}, F_3^{(k)})}{\partial (y, z)} - B_0 \dfrac{\partial (F_1^{(k)}, F_3^{(k)})}{\partial (y, z)} + C_0 \dfrac{\partial (F_1^{(k)}, F_2^{(k)})}{\partial (y, z)} = \\
= \dfrac{\partial (F_1^{(k)}, F_2^{(k)}, F_3^{(k)})}{\partial (x, y, z)} x' + \text{terms of a degree} \geqq 2 \text{ in } x', y', \text{ and } z', \\
- A_0 \dfrac{\partial (F_2^{(k)}, F_3^{(k)})}{\partial (x, z)} + B_0 \dfrac{\partial (F_1^{(k)}, F_3^{(k)})}{\partial (x, z)} - C_0 \dfrac{\partial (F_1^{(k)}, F_2^{(k)})}{\partial (x, z)} = \\
- \dfrac{\partial (F_1^{(k)}, F_2^{(k)}, F_3^{(k)})}{\partial (x, y, z)} y' + \text{terms of a degree} \geqq 2 \text{ in } x', y', \text{ and } z', \\
A_0 \dfrac{\partial (F_2^{(k)}, F_3^{(k)})}{\partial (x, y)} - B_0 \dfrac{\partial (F_1^{(k)}, F_3^{(k)})}{\partial (x, y)} + C_0 \dfrac{\partial (F_1^{(k)}, F_2^{(k)})}{\partial (x, y)} = \\
= \dfrac{\partial (F_1^{(k)}, F_2^{(k)}, F_3^{(k)})}{\partial (x, y, z)} z' + \text{terms of a degree} \geqq 2 \text{ in } x', y', \text{ and } z'.
\end{array}\right. \tag{7}
$$

We state that

$$
x' \equiv 0,\ y' \equiv 0,\ z' \equiv 0 \pmod{p^{\mu-k}}. \tag{8}
$$

The left member of relations (7) is $\equiv 0 \pmod{p^{\mu}}$. If $\mu \leqq 2k+1$, then (8) holds according to (4).

If $\mu \geqq 2k+2$, then all terms of a degree $\geqq 2$ in x', y', and z' are divisible by p^{2k+2} according to (4). Thus according to (3) and (7) x', y', and z' are divisible by p^{k+2}. Hence (8) holds if $\mu \leqq 2k+2$.

If $\mu \geqq 2k+3$ then, according to what we have just proved, all terms of a degree $\geqq 2$ in x', y', and z' are divisible by p^{2k+4}. Thus according to (3) and (7) x', y', and z' are divisible by p^{k+3}. We thus conclude that (8) holds if $\mu \leqq 2k+3$.

In this way (8) follows by induction.

According to (8) A_j, B_j, and C_j $(j \geqq 1)$ are divisible by $p^{\mu-k}$. We now choose $\nu \geqq \mu$. It follows from (6) that A_0, B_0, and C_0 are divisible by $p^{\mu+1}$. However, this contradicts our postulate. Hence

$$
A_0 = B_0 = C_0 = 0\ (p).
$$

Thus μ increases beyond all limits which implies, according to (8), that

$$
x' = y' = z' = 0\ (p).
$$

Since x', y', and z' are rational numbers this means that

$$x' \quad y' \quad z' \quad 0$$

in the usual sense.

The congruence relations (4) thus necessarily imply $x=a$, $y=b$, and $z=c$. If x belongs to one residue class modulo p, y to another and z to a third, there is thus only one solution of (2). The total number of solutions of (2) is $\leqq p^{3(k+1)}$.

We have thus proved

Theorem 3. *Let p be a rational prime and let*

$$F_r(x, y, z) = \sum_{i=0}^{\infty} p^i f_{ir}(x, y, z), \qquad (r=1, 2, 3)$$

where $f_{ir}(x, y, z)$ are polynomials with coefficients which are rational integers modulo p. If we have a k such that the congruence relations

$$F_r^{(k)} \equiv 0 \ (r=1, 2, 3), \ \frac{\partial (F_1^{(k)}, F_2^{(k)}, F_3^{(k)})}{\partial (x, y, z)} \equiv 0 \pmod{p^{k+1}}$$

are not solvable simultaneously, then the system of p-adic equations

$$F_1 = 0, \ F_2 = 0, \ F_3 = 0 \ (p)$$

has only a finite number of solutions in rational integers x, y, and z.

Obviously Theorem 2 can be generalized in this way for any number of variables. Skolem [3, Satz 11] has generalized Theorem 2 in another way.

CHAPTER II

A theorem on a system of independent units of $\boldsymbol{R}(\sqrt[5]{M})$

3. Let M be a rational integer. Without loss of generality we may suppose that $M>1$ and is not divisible by the fifth power of a rational prime. By $\theta=\sqrt[5]{M}$ we denote the real fifth root of M.

Let us consider the ring $\boldsymbol{R}(\theta)$ consisting of all numbers of the form

$$\vartheta=\sum_{i=0}^{4} a_i\,\theta^i,$$

where a_i are rational integers. If $\varepsilon=u+v\,\theta$ is a positive unit of $\boldsymbol{R}(\theta)$ then we get for the norm of ε, $N(\varepsilon)=u^5+M\,v^5=1$. The problem of solving the equation

$$u^5+M\,v^5=1 \tag{1}$$

in rational integers u and v is evidently equivalent to determining all positive units of the form ε in $\boldsymbol{R}(\theta)$. We neglect the trivial solution $u=1,\ v=0$.

It follows from Dirichlet's theorem on the number of fundamental units in an algebraic field that a system of fundamental units in $\boldsymbol{R}(\theta)$ consists of two units. In $\boldsymbol{R}(\theta)$ at most two units are thus independent of each other.

We shall prove the following proposition:

Theorem 4. *If* (1) *has two non-trivial solutions then the corresponding units of* $\boldsymbol{R}(\theta)$ *form a system of independent units of* $\boldsymbol{R}(\theta)$.

Let $u_1=A,\ v_1=B$, and $u_2=C,\ v_2=D$ be two non-trivial solutions of (1) and let

$$\varepsilon_1=A+B\,\theta,\ \varepsilon_2=C+D\,\theta \tag{2}$$

denote the corresponding units of $\boldsymbol{R}(\theta)$. ν_1 and ν_2 denote rational integers. According to Theorem 4 a relation of the form

$$\varepsilon_1^{\nu_1}\,\varepsilon_2^{\nu_2}=1$$

can hold only if $v_1 = v_2 = 0$. If, on the other hand, ε_1 and ε_2 were dependent units of $\boldsymbol{R}(\theta)$, then a relation $\varepsilon_1^X = \varepsilon_2^Y$ would exist, where X and Y were rational integers $\neq 0$. For an arbitrary integer h, $\varepsilon_1^{hX} = \varepsilon_2^{hY}$, would then hold, which implies that the equation $\varepsilon_1^x = \varepsilon_2^y$ would have an infinite number of solutions in rational integers x and y. Similarly, the equation

$$\varepsilon_1^{5x} = \varepsilon_2^{5y} \tag{3}$$

would have an infinite number of solutions in rational integers x and y.

With the aid of the p-adic method we shall prove that (3) only holds for a finite number of rational integers x and y. This contradiction proves the theorem.

4. By (2) we get

$$N(\varepsilon_1) = A^5 + M B^5 = 1, \quad N(\varepsilon_2) = C^5 + M D^5 = 1, \tag{4}$$

and

$$\varepsilon_1^5 = 1 + 5\xi_0, \quad \varepsilon_2^5 = 1 + 5\eta_0,$$

where

$$\xi_0 = A^4 B\theta + 2A^3 B^2 \theta^2 + 2A^2 B^3 \theta^3 + A B^4 \theta^4, \tag{5}$$

$$\eta_0 = C^4 D\theta + 2C^3 D^2 \theta^2 + 2C^2 D^3 \theta^3 + C D^4 \theta^4. \tag{6}$$

We put

$$A = 5^\alpha a, \quad B = 5^\beta b, \quad C = 5^\gamma c, \quad D = 5^\delta d,$$

where a, b, c, and d are rational integers not divisible by 5 and where α, β, γ, and δ are rational integers $\geqq 0$.

According to (4) we have

$$(A, B) = 1, \quad (C, D) = 1. \tag{7}$$

It follows from (7) that at least one of the numbers α and β and at least one of the numbers γ and δ is zero. It follows from (5) and (6) that ξ_0 and η_0 are divisible by $5^{\alpha+\beta}$ and $5^{\gamma+\delta}$ respectively. We may therefore write

$$\xi_0 = 5^{\alpha+\beta}\xi, \quad \eta_0 = 5^{\gamma+\delta}\eta.$$

By p_{rn} and q_{rn} we understand the coefficient of θ^r $(r=0, 1, 2, 3, 4)$ in ξ^n and η^n.

It is easily seen that it is necessary and sufficient to treat the following eight cases.

$$1^\circ \quad \alpha=\beta=\gamma=\delta=0;$$

$$2^\circ \quad \alpha>\gamma\geqq 0, \quad \beta=\delta=0;$$

$$3^\circ \quad \alpha=\gamma>0, \quad \beta=\delta=0;$$

$$4^\circ \quad \alpha>\delta\geqq 0, \quad \beta=\gamma=0;$$

$$5^\circ \quad \delta>\alpha\geqq 0, \quad \beta=\gamma=0;$$

$$6^\circ \quad \alpha=\delta>0, \quad \beta=\gamma=0;$$

$$7^\circ \quad \beta>\delta\geqq 0, \quad \alpha=\gamma=0;$$

$$8^\circ \quad \beta=\delta>0, \quad \alpha=\gamma=0.$$

5. We begin by considering the case $\alpha=\beta=\gamma=\delta=0$. We have $\varepsilon_1=a+b\,\theta$ and $\varepsilon_2=c+d\,\theta$. From

$$\varepsilon_1^{5^x}=(1+5\,\xi)^x=\sum_{k_1=0}^{\infty}\binom{x}{k_1}\xi^{k_1}5^{k_1},\quad \varepsilon_2^{5^y}=(1+5\,\eta)^y=\sum_{k_1=0}^{\infty}\binom{y}{k_1}\eta^{k_1}5^{k_1}$$

we get

$$(8)\qquad \frac{1}{5}(\varepsilon_1^{5^x}-\varepsilon_2^{5^y})=\sum_{k_1=1}^{\infty}5^{k_1-1}\left\{\binom{x}{k_1}\xi^{k_1}-\binom{y}{k_1}\eta^{k_1}\right\}=\sum_{r=0}^{4}F_r(x,\,y)\cdot\theta^r,$$

where

$$F_r(x,\,y)=\sum_{i=0}^{\infty}5^i f_{ir}(x,\,y),$$

and where f_{ir} are polynomials with coefficients which are rational integers modulo 5. By (3) and (8) we obtain

$$(9)\qquad F_r(x,\,y)=0 \quad (5)$$

for $r=0, 1, 2, 3, 4$.

Suppose first that $a\,d-b\,c\not\equiv 0 \pmod 5$. Since

$$F_2^{(0)}=2\,a^3\,b^2\,x-2\,c^3\,d^2\,y,\quad F_3^{(0)}=2\,a^2\,b^3\,x-2\,c^2\,d^3\,y,$$

we have

$$\frac{\partial (F_2^{(0)}, F_3^{(0)})}{\partial (x, y)} \equiv (a\,b\,c\,d)^2 (a\,d - b\,c) \not\equiv 0 \pmod 5.$$

Hence, by Theorem 2, the system of pentadic equations

$$F_2 = 0, \quad F_3 = 0 \quad (5)$$

has only a finite number of solutions in rational integers x and y.

Now let us suppose that μ is a natural number and let

$$a\,d - b\,c \equiv 0 \pmod{5^\mu}.$$

We first prove

Lemma 1. *Let* $\varepsilon_1 = a + b\,\theta$ *and* $\varepsilon_2 = c + d\,\theta$ *be two positive units of* $\boldsymbol{R}(\theta)$ *and* $a\,b\,c\,d \not\equiv 0 \pmod 5$. *If*

$$a\,d - b\,c \equiv 0 \pmod{5^\mu} \quad (10)$$

then $a \equiv c$, $b \equiv d \pmod{5^\mu}$.

Proof. Since ε_1 and ε_2 are positive units of $\boldsymbol{R}(\theta)$ we have

$$a^5 + M\,b^5 = 1, \quad (11)$$

$$c^5 + M\,d^5 = 1, \quad (12)$$

and therefore

$$M \cdot [(a\,d)^5 - (b\,c)^5] = 1 - c^5 - M\,b^5. \quad (13)$$

Let $\mu = 1$. It follows from (13) that $c^5 + M\,b^5 \equiv 1 \pmod 5$. According to (11) we get $a^5 \equiv c^5$ and $a \equiv c \pmod 5$, and by (10) we get $b \equiv d \pmod 5$. Thus our proposition holds if $\mu = 1$.

Let us suppose that the Lemma is proved for $\mu = n$. We shall prove that it holds for $\mu = n + 1$.

According to the hypothesis we may put

$$a = c + r_1 5^n, \quad b = d + r_2 5^n, \quad (14)$$

where r_1 and r_2 are integers. From (11) we get

$$(c + 5^n r_1)^5 + M\,(d + 5^n r_2)^5 = 1,$$

$$c^5 + 5\,c^4\,5^n\,r_1 + 10\,c^3\,(5^n\,r_1)^2 + \cdots + M\,d^5 + M\cdot 5\,d^4\,5^n\,r_2 +$$
$$+ M\cdot 10\,d^3\,(5^n\,r_2)^2 + \cdots = 1,$$

and with the aid of (12) we thus get $c^4\,r_1 + M\,d^4\,r_2 \equiv 0 \pmod 5$ and

$$r_1 + M\,r_2 \equiv 0 \pmod 5. \tag{15}$$

Hence, according to (14)

$$a \equiv c - M\,r_2\,5^n, \quad b \equiv d + r_2\,5^n \pmod{5^{n+1}}.$$

If (10) holds for $\mu = n+1$ we thus get $r_2\,(c + M\,d) \equiv 0 \pmod 5$. Since $c + M\,d \equiv 1 \pmod 5$, $r_2 \equiv 0 \pmod 5$ and, according to (15), $r_1 \equiv 0 \pmod 5$. Hence $a \equiv c$, $b \equiv d \pmod{5^{n+1}}$. The Lemma follows by induction.

Let us suppose that

$$\Delta = a\,d - b\,c \begin{cases} \equiv 0 \pmod{5^\mu} \\ \not\equiv 0 \pmod{5^{\mu+1}}. \end{cases} \tag{16}$$

From Lemma 1 it follows that we may put

$$a = c + a_1\,5^\mu, \quad b = d + b_1\,5^\mu, \tag{17}$$

where a_1 and b_1 are integers. (8) gives

$$F_2^{(\mu)}(x,\,y) = \sum_{k_1=1}^{\mu+1} 5^{k_1-1}\left\{\binom{x}{k_1} p_{2k_1} - \binom{y}{k_1} q_{2k_1}\right\}, \tag{18}$$

$$F_3^{(\mu)}(x,\,y) = \sum_{k_1=1}^{\mu+1} 5^{k_1-1}\left\{\binom{x}{k_1} p_{3k_1} - \binom{y}{k_1} q_{3k_1}\right\}. \tag{19}$$

By putting (17) in (18) and (19) we find[1] that

$$F_2^{(\mu)}(x,\,y) = H_2(x,\,y) - 4\,c^2\,d\,\Delta\,x,$$

$$F_3^{(\mu)}(x,\,y) = H_3(x,\,y) + 4\,c\,d^2\,\Delta\,x,$$

where

$$H_2(x,\,y) = \sum_{k_1=1}^{\mu+1} 5^{k_1-1}\left\{\binom{x}{k_1} - \binom{y}{k_1}\right\} q_{2k_1} - (x-y)\left(q_{21} + \sum_{i=2}^{\mu+1} 5^{i-1}\,h_{i2}(x,\,y)\right),$$

[1] Obviously, terms in $F_2^{(\mu)}$ and $F_3^{(\mu)}$ divisible by $5^{\mu+1}$ or higher powers of 5 may be ignored.

and

$$H_3(x, y) = \sum_{k_1=1}^{\mu+1} 5^{k_1-1} \left\{ \binom{x}{k_1} - \binom{y}{k_1} \right\} q_{3k_1} + (x \quad y) \left(q_{31} + \sum_{i=2}^{\mu+1} 5^{i-1} h_{i3}(x, y) \right).$$

Here h_{i2} and h_{i3} are polynomials with coefficients which are rational integers modulo 5. We get

$$\frac{\partial (F_2^{(\mu)}, F_3^{(\mu)})}{\partial (x, y)} \equiv \frac{\partial (H_2, H_3)}{\partial (x, y)} + 16\, c^4 d^4 \Delta \equiv$$

$$\equiv 5 (x-y) \cdot G(x, y) + 16\, c^4 d^4 \Delta \pmod{5^{\mu+1}},$$

where $G(x, y)$ is a polynomial with coefficients which are rational integers modulo 5.

Let us consider the congruence system

$$F_2^{(\mu)} \equiv 0, \quad F_3^{(\mu)} \equiv 0, \quad 5 (x-y) \cdot G(x, y) + 16\, c^4 d^4 \Delta \equiv 0 \pmod{5^{\mu+1}}. \tag{20}$$

If the first congruence in (20) is to be solvable, then the congruence

$$F_2^{(\mu)} \equiv 0 \pmod{5^{\mu}}$$

must be solvable. We have

$$F_2^{(\mu)}(x, y) \equiv H_2(x, y) \equiv (x-y) \left(q_{21} + \sum_{i=2}^{\mu} 5^{i-1} h_{i2}(x, y) \right) \equiv 0 \pmod{5^{\mu}},$$

and since $q_{21} = 2\, c^3 d^2 \not\equiv 0 \pmod 5$, then

$$x - y \equiv 0 \pmod{5^{\mu}}.$$

Hence we get from (20)

$$16\, c^4 d^4 \Delta \equiv 0 \pmod{5^{\mu+1}},$$

but this is impossible according to (16). Thus the congruence relations (20) are not simultaneously solvable and, according to Theorem 2, the system of pentadic equations

$$F_2 = 0, \quad F_3 = 0 \tag{5}$$

has only a finite number of integral solutions x and y.

6. $\alpha > \gamma \geqq 0, \quad \beta = \delta = 0.$

We have $\varepsilon_1 = 5^\alpha a + b\,\theta$ and $\varepsilon_2 = 5^\gamma c + d\,\theta$. We get

$$\varepsilon_1^5 = 1 + 5^{\alpha+1}\,\xi, \quad \varepsilon_2^5 = 1 + 5^{\gamma+1}\,\eta,$$

where

$$\xi = 5^{3\alpha}\,a^4\,b\,\theta + 2\cdot 5^{2\alpha}\,a^3\,b^2\,\theta^2 + 2\cdot 5^\alpha\,a^2\,b^3\,\theta^3 + a\,b^4\,\theta^4,$$

and

$$\eta = 5^{3\gamma}\,c^4\,d\,\theta + 2\cdot 5^{2\gamma}\,c^3\,d^2\,\theta^2 + 2\cdot 5^\gamma\,c^2\,d^3\,\theta^3 + c\,d^4\,\theta^4.$$

From

$$\varepsilon_1^{5x} = \sum_{k_1=0}^{\infty}\binom{x}{k_1}\,\xi^{k_1}\,5^{k_1(\alpha+1)}, \quad \varepsilon_2^{5y} = \sum_{k_2=0}^{\infty}\binom{y}{k_2}\,\eta^{k_2}\,5^{k_2(\gamma+1)}$$

we get

$$\frac{\varepsilon_2^{5y} - \varepsilon_1^{5x}}{5^{\gamma+1}} = y\,\eta + \binom{y}{2}\,\eta^2\,5^{\gamma+1} + \cdots + \binom{y}{m}\,\eta^m\,5^{(m-1)(\gamma+1)} - \tag{21}$$

$$- x\,\xi\,5^{\alpha-\gamma} + \cdots - \binom{x}{2}\,\xi^2\,5^{2(\alpha+1)-(\gamma+1)} + \cdots - \binom{x}{3}\,\xi^3\,5^{3(\alpha+1)-(\gamma+1)} +$$

$$+ \cdots = \sum_{r=0}^{4} F_r(x, y)\cdot\theta^r,$$

where $m = \left[\frac{\alpha+1}{\gamma+1}\right]$. From (3) and (21) it follows that

$$F_r(x, y) = 0 \pmod 5 \tag{22}$$

for $r = 0, 1, 2, 3, 4$. We find that

$$\frac{F_2(x, y)}{5^{2\gamma}} = 2\,c^3\,d^2\,y + \binom{y}{2}\,c^3\,d^2\,(5^{5\gamma}\,c^5 + 4\,M\,d^5)\cdot 5 + \binom{y}{3}\,q_{23}\,5^2 + \cdots +$$

$$+ \binom{y}{m}\,q_{2m}\,5^{(m-1)(\gamma+1)-2\gamma} - 2\,a^3\,b^2\,x\cdot 5^{3(\alpha-\gamma)} + \cdots -$$

$$- \binom{x}{2}\,a^3\,b^2\,(5^{5\alpha}\,a^5 + 4\,M\,b^5)\,5^{3(\alpha-\gamma)-1} + \cdots - \binom{x}{3}\,p_{23}\,5^{3(\alpha-\gamma)-2} + \cdots.$$

Hence

$$F_{21}(x, y) = \frac{F_2(x, y)}{5^{2\gamma}} = \sum_{i=0}^{\infty} 5^i\,f_{i21}(x, y), \tag{23}$$

where f_{i21} are polynomials with coefficients which are rational integers modulo 5.

Further we get

$$\frac{F_3(x, y)}{5^\gamma} = 2c^2 d^3 y + \binom{y}{2} q_{32} 5 + \cdots + \binom{y}{m} q_{3m} 5^{(m-1)(\gamma+1)\cdot\gamma} -$$

$$- 2a^2 b^3 x \cdot 5^{2(\alpha-\gamma)} + \cdots - \binom{x}{2} p_{32} 5^{2(\alpha-\gamma)+1} + \cdots.$$

Hence

$$F_{31}(x, y) = \frac{F_3(x, y)}{5^\gamma} = \sum_{i=0}^{\infty} 5^i f_{i31}(x, y), \tag{24}$$

where f_{i31} are polynomials with coefficients which are rational integers modulo 5.

From (22), (23), and (24) it follows that

$$F_{21} = 0, \quad F_{31} = 0 \quad (5). \tag{25}$$

However, for $k = 2(\alpha - \gamma)$ we get

$$\frac{\partial(F_{21}^{(k)}, F_{31}^{(k)})}{\partial(x, y)} \equiv 4a^2 b^3 c^3 d^2 5^k \equiv 0 \pmod{5^{k+1}}$$

and, according to Theorem 2, system (25) has only a finite number of integral solutions x and y.

7. $$\alpha = \gamma > 0, \quad \beta = \delta = 0.$$

We have $\varepsilon_1 = 5^\alpha a + b\theta$ and $\varepsilon_2 = 5^\alpha c + d\theta$. We get

$$\frac{\varepsilon_1^{5^x} - \varepsilon_2^{5^y}}{5^{\alpha+1}} \equiv \sum_{k_1=1}^{\infty} 5^{(k_1-1)(\alpha+1)} \left\{ \binom{x}{k_1} \xi^{k_1} - \binom{y}{k_1} \eta^{k_1} \right\} \equiv \sum_{r}^{4} F_r(x, y) \cdot \theta^r. \tag{26}$$

From (3) and (26) we get

$$F_r(x, y) = 0 \quad (5) \tag{27}$$

for $r = 0, 1, 2, 3, 4$. We have

$$F_2(x, y) = 2a^3 b^2 x \cdot 5^{2\alpha} - 2c^3 d^2 y \cdot 5^{2\alpha} + 5^{\alpha+1}\left\{ \binom{x}{2} p_{22} - \binom{y}{2} q_{22} \right\} + \tag{28}$$

$$+ 5^{2(\alpha+1)}\left\{ \binom{x}{3} p_{23} - \binom{y}{3} q_{23} \right\} + \cdots$$

(29) $$F_3(x, y) = \\ = 2a^2b^3x \cdot 5^\alpha - 2c^2d^3y \cdot 5^\alpha + 5^{\alpha+1}\left\{\binom{x}{2}p_{32} - \binom{y}{2}q_{32}\right\} + \cdots$$

Since p_{22} and q_{22} are divisible by 5^α we may write

(30) $$F_{21}(x, y) = \frac{F_2(x, y)}{5^{2\alpha}} = \sum_{i=0}^{\infty} 5^i f_{i21}(x, y),$$

(31) $$F_{31}(x, y) = \frac{F_3(x, y)}{5^\alpha} = \sum_{i=0}^{\infty} 5^i f_{i31}(x, y),$$

where f_{i21} and f_{i31} are polynomials with coefficients which are rational integers modulo 5. From (27) we get

(32) $$F_{21} = 0, \quad F_{31} = 0 \quad (5).$$

Let us first suppose that $ad - bc \not\equiv 0 \pmod 5$. Since

$$\frac{\partial(F_{21}^{(0)}, F_{31}^{(0)})}{\partial(x, y)} \equiv (abcd)^2(ad - bc) \not\equiv 0 \pmod 5$$

system (32) has only a finite number of integral solutions x and y according to Theorem 2.

Now let us suppose that μ is a natural number and let

$$ad - bc \equiv 0 \pmod{5^\mu}.$$

We first prove

Lemma 2. *Let* $\varepsilon_1 = 5^\alpha a + b\theta$ *and* $\varepsilon_2 = 5^\alpha c + d\theta$ *be two positive units of* $\boldsymbol{R}(\theta)$, $\alpha > 0$ *and* $abcd \not\equiv 0 \pmod 5$. *If*

$$ad - bc \equiv 0 \pmod{5^\mu}$$

then $a \equiv c$, $b \equiv d \pmod{5^\mu}$.

Proof. Since

(33) $$(5^\alpha a)^5 + Mb^5 = 1, \quad (5^\alpha c)^5 + Md^5 = 1,$$

we have

(34) $$M \cdot 5^{5\alpha}[(ad)^5 - (bc)^5] = 1 - (5^\alpha c)^5 - Mb^5,$$

and since $\alpha > 0$ then $M \not\equiv 0 \pmod 5$.

Hence (34) gives

$$(5^{\alpha}c)^5 + M b^5 \equiv 1 \pmod{5^{5\alpha+\mu}}. \tag{35}$$

(33) and (35) gives

$$a^5 \equiv c^5 \pmod{5^{\mu}}, \quad b^5 \equiv d^5 \pmod{5^{5\alpha+\mu}},$$

or[1]

$$a \equiv c + r_1 5^{\mu-1}, \quad b \equiv d + r_2 5^{\mu-1} \pmod{5^{\mu}},$$

where r_1 and r_2 are integers, $0 \leqq r_1 \leqq 4$ and $0 \leqq r_2 \leqq 4$. Since $\alpha > 0$ we have $r_2 = 0$. If $\mu = 1$ then $r_1 = 0$. However, $r_1 = 0$ even when $\mu > 1$, since

$$a d - b c \equiv d r_1 5^{\mu-1} \equiv 0 \pmod{5^{\mu}}$$

which gives $r_1 \equiv 0 \pmod 5$ and $r_1 = 0$.

Let us suppose that

$$\Delta = a d - b c \begin{cases} \equiv 0 \pmod{5^{\mu}} \\ \not\equiv 0 \pmod{5^{\mu+1}}. \end{cases}$$

According to Lemma 2 we may put

$$a = c + a_1 5^{\mu}, \quad b = d + b_1 5^{\mu}, \tag{36}$$

where a_1 and b_1 are integers.

According to (28)–(31) we may write

$$F_{21}^{(\mu)}(x, y) = \sum_{k_1=1}^{\sigma} 5^{(k_1-1)(\alpha+1)-2\alpha} \left\{ \binom{x}{k_1} p_{2k_1} - \binom{y}{k_1} q_{2k_1} \right\}, \tag{37}$$

$$F_{31}^{(\mu)}(x, y) = \sum_{k_1=1}^{\tau} 5^{(k_1-1)(\alpha+1)-\alpha} \left\{ \binom{x}{k_1} p_{3k_1} - \binom{y}{k_1} q_{3k_1} \right\}, \tag{38}$$

where

$$\sigma = \left[\frac{3\alpha+\mu+1}{\alpha+1}\right] \quad \text{and} \quad \tau = \left[\frac{2\alpha+\mu+1}{\alpha+1}\right].$$

By putting (36) in (37) and (38) we get

$$F_{21}^{(\mu)}(x, y) = H_2(x, y) - 4 c^2 d \Delta x,$$

$$F_{31}^{(\mu)}(x, y) = H_3(x, y) + 4 c d^2 \Delta x,$$

[1] Cp. e.g. HARDY-WRIGHT [1, Theorem 126, p. 96].

where

$$H_2(x, y) - \sum_{k_1=1}^{\sigma} 5^{(k_1-1)(\alpha+1)-2\alpha} \left\{ \binom{x}{k_1} - \binom{y}{k_1} \right\} q_{2k_1},$$

$$H_3(x, y) = \sum_{k_1=1}^{\tau} 5^{(k_1-1)(\alpha+1)-\alpha} \left\{ \binom{x}{k_1} - \binom{y}{k_1} \right\} q_{3k_1}.$$

We may write

$$H_2(x, y) = (x-y)\left(2c^3d^2 + \sum_{k_1=2}^{\sigma} 5^{(k_1-1)(\alpha+1)-2\alpha} h_{k_1 2}(x, y)\right),$$

$$H_3(x, y) = (x-y)\left(2c^2d^3 + \sum_{k_1=2}^{\tau} 5^{(k_1-1)(\alpha+1)-\alpha} h_{k_1 3}(x, y)\right),$$

where $h_{k_1 2}$ and $h_{k_1 3}$ are polynomials with coefficients which are rational integers modulo 5. Further we find

$$\frac{\partial(F_{21}^{(\mu)}, F_{31}^{(\mu)})}{\partial(x, y)} \equiv \frac{\partial(H_2, H_3)}{\partial(x, y)} + 16c^4d^4\Delta = 5(x-y)\cdot G(x, y) + 16c^4d^4\Delta \pmod{5^{\mu+1}},$$

where $G(x, y)$ is a polynomial with coefficients which are rational integers modulo 5.

Let us consider the congruence relations

$$F_{21}^{(\mu)} \equiv 0, \quad F_{31}^{(\mu)} \equiv 0, \quad 5(x-y)\cdot G(x, y) + 16c^4d^4\Delta \equiv 0 \pmod{5^{\mu+1}}.$$

As before (no. **5**) we find that these congruences are not simultaneously solvable. Hence, according to Theorem 2, the system of pentadic equations (32) has only a finite number of integral solutions x and y.

8. $\alpha > \delta \geqq 0, \quad \beta = \gamma = 0.$

We have $\varepsilon_1 = 5^\alpha a + b\theta$ and $\varepsilon_2 = c + 5^\delta d\theta$. We get

$$\varepsilon_1^5 = 1 + 5^{\alpha+1}\xi, \quad \varepsilon_2^5 = 1 + 5^{\delta+1}\eta,$$

where

$$\xi = 5^{3\alpha}a^4b\theta + 2\cdot 5^{2\alpha}a^3b^2\theta^2 + 2\cdot 5^\alpha a^2b^3\theta^3 + ab^4\theta^4,$$

and

$$\eta = c^4d\theta + 2\cdot 5^\delta c^3d^2\theta^2 + 2\cdot 5^{2\delta}c^2d^3\theta^3 + 5^{3\delta}cd^4\theta^4.$$

From

$$\varepsilon_1^{5x} = \sum_{k_1=0}^{\infty} \binom{x}{k_1} \xi^{k_1} 5^{k_1(\alpha+1)}, \quad \varepsilon_2^{5y} = \sum_{k_2=0}^{\infty} \binom{y}{k_2} \eta^{k_2} 5^{k_2(\delta+1)}$$

we get

$$(39)\qquad \frac{\varepsilon_2^{5^y}-\varepsilon_1^{5^x}}{5^{\delta+1}} = y\,\eta + \binom{y}{2}\eta^2\,5^{\delta+1} + \binom{y}{3}\eta^3\,5^{2(\delta+1)} + \cdots +$$

$$+ \binom{y}{m}\eta^m\,5^{(m-1)(\delta+1)} - x\,\xi\,5^{\alpha-\delta} + \cdots - \binom{x}{2}\xi^2\,5^{2(\alpha+1)-(\delta+1)} +$$

$$\cdots - \sum_{r=0}^{4} F_r(x, y)\cdot\theta^r.$$

where $m=\left[\frac{\alpha+1}{\delta+1}\right]$. By (3) and (39) we get

$$(40)\qquad F_r(x, y)=0 \quad (5)$$

for $r=0, 1, 2, 3, 4$. We find that

$$\frac{F_2(x, y)}{5^\delta} = 2\,c^3\,d^2\,y + \binom{y}{2}q_{22}\,5 + \cdots + \binom{y}{m}q_{2m}\,5^{(m-1)(\delta+1)-\delta} -$$

$$-2\,a^3\,b^2\,x\,5^{3\alpha-2\delta}+\cdots.$$

Hence

$$(41)\qquad F_{21}(x, y)=\frac{F_2(x, y)}{5^\delta}=\sum_{t=0}^{\infty}5^t\,f_{t21}(x, y),$$

where f_{t21} are polynomials with coefficients which are rational integers modulo 5.

Further we find that

$$F_3(x, y) = 2\,c^2\,d^3\,y\cdot 5^{2\delta} + \binom{y}{2}c^2\,d^3\,(4\,c^5 + M\cdot 5^{5\delta}\,d^5)\,5^{2\delta+1} +$$

$$+ \binom{y}{3}q_{33}\,5^{2(\delta+1)} + \cdots + \binom{y}{m}q_{3m}\,5^{(m-1)(\delta+1)} - 2\,a^2\,b^3\,x\,5^{2\alpha-\delta} +$$

$$\cdots - \binom{x}{2}\mu_{32}\,5^{2(\alpha-1)-(\delta+1)}\cdots.$$

First we suppose that $2\,\delta \leqq 2\,\alpha-\delta$. We may write

$$(42)\qquad F_{31}(x, y)=\frac{F_3(x, y)}{5^{2\delta}}=\sum_{t=0}^{\infty}5^t\,f_{t31}(x, y),$$

where f_{t31} are polynomials with coefficients which are rational integers modulo 5. From (40), (41) and (42) it follows that

$$(43)\qquad F_{21}=0, \quad F_{31}=0 \quad (5).$$

However, for $k = 2\alpha - 3\delta$ we get

$$\frac{\partial (F_{21}^{(k)}, F_{31}^{(k)})}{\partial (x, y)} \equiv 4 a^2 b^3 c^3 d^2 5^k \not\equiv 0 \pmod{5^{k+1}}$$

and, according to Theorem 2, system (43) has only a finite number of integral solutions x and y.

Now let us suppose that $2\delta > 2\alpha - \delta$. We find that

$$F_{32}(x, y) = \frac{F_3(x, y)}{5^{2\alpha-\delta}} = \sum_{i=0}^{\infty} 5^i f_{i32}(x, y), \tag{44}$$

where f_{i32} are polynomials with coefficients which are rational integers modulo 5. From (40), (41), and (44) we get

$$F_{21} = 0, \quad F_{32} = 0 \quad (5). \tag{45}$$

We get

$$\frac{\partial (F_{21}^{(0)}, F_{32}^{(0)})}{\partial (x, y)} \equiv 4 a^2 b^3 c^3 d^2 \not\equiv 0 \pmod{5}.$$

Hence, by Theorem 2, system (45) has only a finite number of solutions in rational integers x and y.

9. $\delta > \alpha \geqq 0, \quad \beta = \gamma = 0.$

Analogous with (39) we get

$$\frac{\varepsilon_1^{5x} - \varepsilon_2^{5y}}{5^{\alpha+1}} = x\xi + \binom{x}{2} \xi^2 5^{\alpha+1} + \binom{x}{3} \xi^3 5^{2(\alpha+1)} + \cdots + \tag{46}$$

$$+ \binom{x}{m} \xi^m 5^{(m-1)(\alpha+1)} - y\eta 5^{\delta-\alpha} + \cdots - \binom{y}{2} \eta^2 5^{2(\delta+1)-(\alpha+1)} + \cdots -$$

$$- \sum_{r=0}^{4} F_r(x, y) \theta^r,$$

where $m = \left[\frac{\delta+1}{\alpha+1}\right]$. From (3) and (46) we get

$$F_r(x, y) = 0 \quad (5) \tag{47}$$

for $r = 0, 1, 2, 3, 4$. We find that

$$F_2(x, y) = 2a^3b^2x \cdot 5^{2\alpha} + \binom{x}{2} a^3b^2 (5^{5\alpha}a^5 + 4Mb^5) 5^{2\alpha+1} +$$

$$+ \binom{x}{3} p_{23} 5^{2(\alpha+1)} + \cdots + \binom{x}{m} p_{2m} 5^{(m-1)(\alpha+1)} - 2c^3d^2y 5^{2\delta-\alpha} + \cdots -$$

$$- \binom{y}{2} q_{22} 5^{2(\delta+1)-(\alpha+1)} + \cdots .$$

If $2\alpha \leqq 2\delta - \alpha$ we may write

$$F_{21}(x, y) = \frac{F_2(x, y)}{5^{2\alpha}} = \sum_{i=0}^{\infty} 5^i f_{i21}(x, y), \tag{48}$$

and if $2\alpha > 2\delta - \alpha$ we may write

$$F_{22}(x, y) = \frac{F_2(x, y)}{5^{2\delta-\alpha}} = \sum_{i=0}^{\infty} 5^i f_{i22}(x, y), \tag{49}$$

where f_{i21} and f_{i22} are polynomials with coefficients which are rational integers modulo 5. Further we get

$$F_{31}(x, y) = \frac{F_3(x, y)}{5^{\alpha}} = \sum_{i=0}^{\infty} 5^i f_{i31}(x, y), \tag{50}$$

where f_{i31} are polynomials of the same kind. We have

$$F_{31}(x, y) = 2a^2b^3x + \binom{x}{2} p_{32} 5 + \cdots + \binom{x}{m} p_{3m} 5^{(m-1)(\alpha+1)-\alpha} -$$

$$- 2c^2d^3y 5^{3\delta-2\alpha} + \cdots$$

Let us first suppose $2\alpha \leqq 2\delta - \alpha$. From (47), (48) and (50) we get

$$F_{21} \equiv 0, \quad F_{31} \equiv 0 \pmod{5}.$$

According to Theorem 2 this system has only a finite number of integral solutions since

$$\frac{\partial(F_{21}^{(k)}, F_{31}^{(k)})}{\partial(x, y)} \equiv 4a^2b^3c^3d^2 5^k \equiv 0 \pmod{5^{k+1}}$$

for $k = 2\delta - 3\alpha$.

Suppose next that $2\alpha > 2\delta - \alpha$. By (47), (49) and (50) we get

$$F_{22} \equiv 0, \quad F_{31} \equiv 0 \pmod{5},$$

but, according to Theorem 2, this system has only a finite number of solutions in rational integers since

$$\frac{\partial (F_{22}^{(0)}, F_{31}^{(0)})}{\partial (x, y)} \equiv 4\, a^2 b^3 c^3 d^2 \not\equiv 0 \pmod 5.$$

10. $\alpha = \delta > 0, \quad \beta = \gamma = 0.$

We have $\varepsilon_1 = 5^a a + b\,\theta$ and $\varepsilon_2 = c + 5^a d\,\theta$. We get

$$(51) \qquad \frac{\varepsilon_1^{5x} - \varepsilon_2^{5y}}{5^{a+1}} = \sum_{k_1=1}^{\infty} 5^{(k_1-1)(a+1)} \left\{ \binom{x}{k_1} \xi^{k_1} - \binom{y}{k_1} \eta^{k_1} \right\} = \sum_{r=0}^{4} F_r (x, y)\, \theta^r.$$

From (3) and (51) we get

$$(52) \qquad F_r (x, y) = 0 \quad (5)$$

for $r = 0, 1, 2, 3, 4$. We get

$$(53) \qquad F_{21} (x, y) = \frac{F_2 (x, y)}{5^a} = 2\, a^3 b^2 x \cdot 5^a - 2\, c^3 d^2 y +$$

$$+ \sum_{k_1=2}^{\infty} 5^{(k_1-1)(a+1)-a} \left\{ \binom{x}{k_1} p_{2k_1} - \binom{y}{k_1} q_{2k_1} \right\} = \sum_{i=0}^{\infty} 5^i f_{i21} (x, y),$$

$$(54) \qquad F_{31} (x, y) = \frac{F_3 (x, y)}{5^a} = 2\, a^2 b^3 x - 2\, c^2 d^3 y \cdot 5^a +$$

$$- \sum_{k_1=2}^{\infty} 5^{(k_1-1)(a+1)-a} \left\{ \binom{x}{k_1} p_{3k_1} - \binom{y}{k_1} q_{3k_1} \right\} = \sum_{i=0}^{\infty} 5^i f_{i31} (x, y),$$

where f_{i21} and f_{i31} are polynomials with coefficients which are rational integers modulo 5.

(52)—(54) gives

$$F_{21} = 0, \quad F_{31} = 0 \quad (5).$$

However, according to Theorem 2 this system has only a finite number of integral solutions since

$$\frac{\partial (F_{21}^{(0)}, F_{31}^{(0)})}{\partial (x, y)} \equiv 4\, a^2 b^3 c^3 d^2 \not\equiv 0 \pmod 5.$$

11. $\beta > \delta \geqq 0, \quad \alpha = \gamma = 0.$

We have $\varepsilon_1 = a + 5^\beta b\,\theta$ and $\varepsilon_2 = c + 5^\delta d\,\theta$. We get

$$\varepsilon_1^{5^i} = 1 + 5^{i+1}\xi, \quad \varepsilon_2^{5^i} = 1 + 5^{i+1}\eta,$$

where

$$\xi = a^4 b\,\theta + 2\cdot 5^\beta a^3 b^2 \theta^2 + 2\cdot 5^{2\beta} a^2 b^3 \theta^3 + 5^{3\beta} a b^4 \theta^4,$$

and

$$\eta = c^4 d\,\theta + 2\cdot 5^\delta c^3 d^2 \theta^2 + 2\cdot 5^{2\delta} c^2 d^3 \theta^3 + 5^{3\delta} c d^4 \theta^4.$$

By

$$\varepsilon_1^{5^x} = \sum_{k_1=0}^{\infty} \binom{x}{k_1} \xi^{k_1} 5^{k_1(\beta+1)}, \quad \varepsilon_2^{5^y} = \sum_{k_2=0}^{\infty} \binom{y}{k_2} \eta^{k_2} 5^{k_2(\delta+1)}$$

we get

$$(55) \qquad \frac{\varepsilon_2^{5^y} - \varepsilon_1^{5^x}}{5^{\delta+1}} = y\,\eta + \binom{y}{2}\eta^2 5^{\delta-1} + \cdots + \binom{y}{m}\eta^m 5^{(m-1)(\delta+1)} -$$

$$- x\,\xi\,5^{\beta-\delta} + \cdots - \binom{x}{2}\xi^2 5^{2(\beta+1)-(\delta+1)} + \cdots -$$

$$- \binom{x}{3}\xi^3 5^{3(\beta+1)-(\delta+1)} - \cdots - \sum_{r=0}^{4} F_r(x, y)\,\theta^r,$$

where $m = \left[\frac{\beta+1}{\delta+1}\right]$. From (3) and (55) we get

$$(56) \qquad F_r(x, y) = 0 \pmod 5$$

for $r = 0, 1, 2, 3, 4$.

We find

$$(57) \qquad F_{21}(x, y) = \frac{F_2(x, y)}{5^\delta} = 2\,c^3 d^2 y + \binom{y}{2} q_{22}\,5 + \cdots +$$

$$+ \binom{y}{m} q_{2m}\,5^{(m-1)(\delta+1)-\delta} - 2\,a^3 b^2 x\,5^{2(\beta-\delta)} + \cdots -$$

$$- \binom{x}{2} p_{22}\,5^{2(\beta-\delta)+1} + \cdots = \sum_{i=0}^{\infty} 5^i f_{i21}(x, y),$$

$$(58) \qquad F_{31}(x, y) = \frac{F_3(x, y)}{5^{2\delta}} = 2\,c^2 d^3 y + \binom{y}{2} c^2 d^3 (4\,c^5 + M\cdot 5^{5\delta} d^5)\,5 +$$

$$+ \binom{y}{3} q_{33}\,5^2 + \cdots + \binom{y}{m} q_{3m}\,5^{(m-1)(\delta+1)-2\delta} - 2\,a^2 b^3 x\,5^{3(\beta-\delta)} + \cdots -$$

$$-\binom{x}{2} a^2 b^3 (4 a^5 + M \cdot 5^{5\beta} b^5) 5^{3(\beta-\delta)+1} + \cdots - \binom{x}{3} p_{33} 5^{3(\beta-\delta)+2} +$$

$$+ \cdots = \sum_{i=0}^{\infty} 5^i f_{i31}(x, y),$$

where f_{i21} and f_{i31} are polynomials with coefficients which are rational integers modulo 5. It follows from (56), (57) and (58) that

$$F_{21} = 0, \quad F_{31} = 0 \quad (5) \tag{59}$$

Since

$$\frac{\partial (F_{21}^{(k)}, F_{31}^{(k)})}{\partial (x, y)} \equiv -4 a^3 b^2 c^2 d^3 5^k \not\equiv 0 \pmod{5^{k+1}}$$

for $k = 2(\beta - \delta)$ the system of pentadic equations (59) has only a finite number of integral solutions according to Theorem 2.

12. $\beta = \delta > 0, \quad \alpha = \gamma = 0.$

We have $\varepsilon_1 = a + 5^\beta b\,\theta$ and $\varepsilon_2 = c + 5^\beta d\,\theta$. We get

$$\frac{\varepsilon_1^{5^x} - \varepsilon_2^{5^y}}{5^{\beta+1}} = \sum_{k_1=1}^{\infty} 5^{(k_1-1)(\beta+1)} \left\{ \binom{x}{k_1} \xi^{k_1} - \binom{y}{k_1} \eta^{k_1} \right\} = \tag{60}$$

$$= \sum_{r=0}^{4} F_r(x, y) \cdot \theta^r.$$

It follows from (3) and (60) that

$$F_r(x, y) = 0 \quad (5) \tag{61}$$

for $r = 0, 1, 2, 3, 4$.

From (60) we get

$$F_{21}(x, y) = \frac{F_2(x, y)}{5^\beta} = 2 a^3 b^2 x - 2 c^3 d^2 y + \tag{62}$$

$$+ \sum_{k_1=2}^{\infty} 5^{(k_1-1)(\beta+1)-\beta} \left\{ \binom{x}{k_1} p_{2k_1} - \binom{y}{k_1} q_{2k_1} \right\} = \sum_{i=0}^{\infty} 5^i f_{i21}(x, y),$$

$$F_{31}(x, y) = \frac{F_3(x, y)}{5^{2\beta}} = 2 a^2 b^3 x - 2 c^2 d^3 y + \tag{63}$$

$$+ \sum_{k_1=2}^{\infty} 5^{(k_1-1)(\beta+1)-2\beta} \left\{ \binom{x}{k_1} p_{3k_1} - \binom{y}{k_1} q_{3k_1} \right\} - \sum_{i=0}^{\infty} 5^i f_{i31}(x, y).$$

Since p_{32} and q_{32} are divisible by 5^β then f_{i21} and f_{i31} are polynomials with coefficients which are rational integers modulo 5. It follows from (61), (62) and (63) that

$$F_{21} \equiv 0, \quad F_{31} \equiv 0 \quad (5). \tag{64}$$

We have

$$\frac{\partial (F_{21}^{(0)}, F_{31}^{(0)})}{\partial (x, y)} \equiv (a b c d)^2 (a d - b c) \equiv 0 \pmod 5.$$

Hence, according to Theorem 2, system (64) has at most a finite number of integral solutions x and y when $a d - b c \not\equiv 0 \pmod 5$.

Let us now suppose that $a d - b c \equiv 0 \pmod{5^\mu}$ where μ is a natural number. We first prove

Lemma 3. *Let* $\varepsilon_1 = a + 5^\beta b \theta$ *and* $\varepsilon_2 = c + 5^\beta d \theta$ *be two positive units of* $\boldsymbol{R}(\theta)$, $\beta > 0$ *and* $a b c d \not\equiv 0 \pmod 5$. *If*

$$a d - b c \equiv 0 \pmod{5^\mu}$$

then $a \equiv c$, $b \equiv d \pmod{5^\mu}$.

Proof. Let M be divisible by 5^s but not by 5^{s+1}. (We may suppose that $s = 0, 1, 2, 3$ or 4.) It is easily seen (Cp. the proof of Lemma 2, no. **7**) that

$$M \cdot 5^{5\beta} [(a d)^5 - (b c)^5] = 1 - c^5 - M (5^\beta b)^5$$

and hence

$$c^5 + M (5^\beta b)^5 \equiv 1 \pmod{5^{s+5\beta+\mu}}.$$

We get

$$a^5 \equiv c^5 \pmod{5^{s+5\beta+\mu}}, \quad b^5 \equiv d^5 \pmod{5^\mu}$$

and

$$a \equiv c + r_1 5^{\mu-1}, \quad b \equiv d + r_2 5^{\mu-1} \pmod{5^\mu},$$

where r_1 and r_2 are integers, $0 \leqq r_1 \leqq 4$ and $0 \leqq r_2 \leqq 4$. Since $\beta > 0$ $r_1 = 0$. If $\mu = 1$ then $r_2 = 0$. However, $r_2 = 0$ even when $\mu > 1$, since

$$a d - b c \equiv - c r_2 5^{\mu-1} \equiv 0 \pmod{5^\mu},$$

which gives $r_2 \equiv 0 \pmod 5$ and $r_2 = 0$.

Let us suppose that

$$\Delta = a\,d - b\,c \begin{cases} \equiv 0 \pmod{5^{\mu}} \\ \not\equiv 0 \pmod{5^{\mu+1}}. \end{cases} \tag{65}$$

According to Lemma 3 we may put

$$a = c + a_1 5^{\mu}, \quad b = d + b_1 5^{\mu}, \tag{66}$$

where a_1 and b_1 are integers. From (62) and (63) we get

$$F_{21}^{(\mu)}(x, y) = \sum_{k_1=1}^{\sigma} 5^{(k_1-1)(\beta+1)-\beta} \left\{ \binom{x}{k_1} p_{2k_1} - \binom{y}{k_1} q_{2k_1} \right\}, \tag{67}$$

$$F_{31}^{(\mu)}(x, y) = \sum_{k_1=1}^{\tau} 5^{(k_1-1)(\beta+1)-2\beta} \left\{ \binom{x}{k_1} p_{3k_1} - \binom{y}{k_1} q_{3k_1} \right\}, \tag{68}$$

where

$$\sigma = \left[\frac{2\beta + \mu + 1}{\beta + 1}\right] \quad \text{and} \quad \tau = \left[\frac{3\beta + \mu + 1}{\beta + 1}\right].$$

Putting (66) in (67) and (68) we get

$$F_{21}^{(\mu)}(x, y) = H_2(x, y) - 4\,c^2\,d\,\Delta\,x,$$

$$F_{31}^{(\mu)}(x, y) = H_3(x, y) + 4\,c\,d^2\,\Delta\,x,$$

where

$$H_2(x, y) = \sum_{k_1=1}^{\sigma} 5^{(k_1-1)(\beta+1)-\beta} \left\{ \binom{x}{k_1} - \binom{y}{k_1} \right\} q_{2k_1},$$

and

$$H_3(x, y) = \sum_{k_1=1}^{\tau} 5^{(k_1-1)(\beta+1)-2\beta} \left\{ \binom{x}{k_1} - \binom{y}{k_1} \right\} q_{3k_1}.$$

We may write

$$H_2(x, y) = (x-y)\left(2\,c^3\,d^2 + \sum_{k_1=2}^{\sigma} 5^{(k_1-1)(\beta+1)-\beta} h_{k_1 2}(x, y)\right),$$

$$H_3(x, y) = (x-y)\left(2\,c^2\,d^3 + \sum_{k_1=2}^{\tau} 5^{(k_1-1)(\beta+1)-2\beta} h_{k_1 3}(x, y)\right),$$

where $h_{k_1 2}$ and $h_{k_1 3}$ are polynomials with coefficients which are rational integers modulo 5. As before we find

$$\frac{\partial(F_{21}^{(\mu)}, F_{31}^{(\mu)})}{\partial(x, y)} \equiv 5\,(x-y)\,G(x, y) + 16\,c^4\,d^4\,\Delta \pmod{5^{\mu+1}},$$

where $G(x, y)$ is a polynomial with coefficients which are rational integers modulo 5. As before we find that

$$\frac{\partial (F_{21}^{(\mu)}, F_{31}^{(\mu)})}{\partial (x, y)} \not\equiv 0 \pmod{5^{\mu+1}}.$$

Hence system (64) has only a finite number of integral solutions x and y.

Thus we have furnished a complete proof of the theorem formulated at the beginning of this chapter.

CHAPTER III

An upper limit to the number of integral solutions of $u^5 + Mv^5 = 1$

§ 1. Introduction

13. Let us assume that the equation

$$u^5 + Mv^5 = 1 \tag{1}$$

has three non-trivial solutions

$$\begin{cases} u_1 = A, \\ v_1 = B, \end{cases} \quad \begin{cases} u_2 = C, \\ v_2 = D, \end{cases} \quad \begin{cases} u_3 = F, \\ v_3 = G. \end{cases}$$

$\boldsymbol{R}(\theta)$ has therefore three positive units

$$\varepsilon_1 = A + B\theta, \ \varepsilon_2 = C + D\theta, \ \varepsilon_3 = F + G\theta. \tag{2}$$

According to our result in Chapter II these units taken two by two compose systems of independent units of $\boldsymbol{R}(\theta)$. Between the three units there must exist a relation of the form

$$\varepsilon_1^{\nu_1} \varepsilon_2^{\nu_2} \varepsilon_3^{\nu_3} = 1, \tag{3}$$

where ν_1, ν_2 and ν_3 are rational integers different from zero. If one of the numbers ν_1, ν_2 and ν_3 were equal to zero the others would also be equal to zero according to Theorem 4. But $\nu_1 \nu_2 \nu_3 \neq 0$ for if it followed from (3) that $\nu_1 = \nu_2 = \nu_3 = 0$ this would imply that there existed systems of three independent units in $\boldsymbol{R}(\theta)$ which would contradict DIRICHLET's theorem.

We may write (3)

$$\varepsilon_1^x \varepsilon_2^y = \varepsilon_3^z, \qquad (xyz \neq 0) \tag{4}$$

Let h be any rational integer. If (4) holds, then the relation $\varepsilon_1^{hx}\varepsilon_2^{hy}=\varepsilon_3^{hz}$ also holds. Thus (4) has an infinite number of solutions in rational integers x, y and z. Similarly, the equation

$$\varepsilon_1^{5x}\varepsilon_2^{5y}=\varepsilon_3^{5z} \qquad (xyz\neq 0) \tag{5}$$

has an infinite number of solutions in rational integers x, y and z.

In the subsequent argument we shall make use of the p-adic method to prove that (5) is valid only for a finite number of rational integers. Thus we have a contradiction. This must therefore imply that (1) has at most two non-trivial solutions in rational integers u and v.

We distinguish between various cases according to whether A, B, C, D, F and G are divisible by 5 or not. We put

$$A=5^{\alpha}a,\ B=5^{\beta}b,\ C=5^{\gamma}c,\ D=5^{\delta}d,\ F=5^{\varkappa}f,\ G=5^{\lambda}g, \tag{6}$$

where a, b, c, d, f and g are rational integers not divisible by 5 and where α, β, γ, δ, $\varkappa$ and λ are rational integers $\geqq 0$.

It follows from (1) that

$$(A, B)=1,\ \ (C, D)=1,\ \ (F, G)=1. \tag{7}$$

According to (7) and since ε_1, ε_2 and ε_3 are permutable it is sufficient to treat the following five cases

1° $\alpha=\beta=\gamma=\delta=\varkappa=\lambda=0$;

2° $\beta=\delta=\lambda=0$, one, two or three of the numbers α, γ and $\varkappa\neq 0$;

3° $\beta=\gamma=\lambda=0$, one, two or three of the numbers α, δ and $\varkappa\neq 0$;

4° $\beta=\gamma=\varkappa=0$, one, two or three of the numbers α, δ and $\lambda\neq 0$;

5° $\alpha=\gamma=\varkappa=0$, one, two or three of the numbers β, δ and $\lambda\neq 0$.

§ 2. $u^5 + M v^5 = 1$ cannot have three integral solutions $u_1; v_1, u_2; v_2$ and $u_3; v_3$ such that $u_j v_j \not\equiv 0 \pmod 5$ for $j = 1, 2, 3$.

14. In this paragraph we shall treat the case

$$\alpha = \beta = \gamma = \delta = \varkappa = \lambda = 0.$$

We have

$$\varepsilon_1 = a + b\,\theta, \quad \varepsilon_2 = c + d\,\theta, \quad \varepsilon_3 = f + g\,\theta,$$

and

$$\varepsilon_1^5 = 1 + 5\,\xi, \quad \varepsilon_2^5 = 1 + 5\,\eta, \quad \varepsilon_3^5 = 1 + 5\,\zeta, \tag{1}$$

where

$$\xi = a^4 b\,\theta + 2\,a^3 b^2\,\theta^2 + 2\,a^2 b^3\,\theta^3 + a\,b^4\,\theta^4,$$

$$\eta = c^4 d\,\theta + 2\,c^3 d^2\,\theta^2 + 2\,c^2 d^3\,\theta^3 + c\,d^4\,\theta^4,$$

$$\zeta = f^4 g\,\theta + 2\,f^3 g^2\,\theta^2 + 2\,f^2 g^3\,\theta^3 + f\,g^4\,\theta^4.$$

We denote the coefficient of θ^r $(r = 0, 1, 2, 3, 4)$ in ξ^n, η^n, ζ^n and $\xi^m \eta^n$ by p_{rn}, q_{rn}, r_{rn} and s_{rmn} respectively.

It follows from (1) that

$$\frac{1}{5}(\varepsilon_1^{5x}\,\varepsilon_2^{5y} - \varepsilon_3^{5z}) = \sum_{\nu=1}^{\infty} 5^{\nu-1} \left\{ \sum_{s=0}^{\nu} \left[\binom{x}{\nu - s} \binom{y}{s} \xi^{\nu - s} \eta^s \right] - \binom{z}{\nu} \zeta^\nu \right\} = \tag{2}$$

$$= \sum_{r=0}^{4} F_r(x, y, z) \cdot \theta^r,$$

where

$$F_r(x, y, z) = \sum_{i=0}^{\infty} 5^i f_{ir}(x, y, z), \tag{3}$$

and where f_{ir} are polynomials with coefficients which are rational integers modulo 5.

If

$$\varepsilon_1^{5x}\,\varepsilon_2^{5y} = \varepsilon_3^{5z} \tag{4}$$

then

$$F_r(x, y, z) = 0 \quad (5)$$

for $r = 0, 1, 2, 3, 4$.

We get

$$(6)\quad \begin{cases} F_1 = p_{11}\,x + q_{11}\,y - r_{11}\,z + \\ \qquad + 5\left\{\binom{x}{2} p_{12} + x\,y\,s_{111} + \binom{y}{2} q_{12} - \binom{z}{2} r_{12}\right\} + \cdots, \\ F_2 = p_{21}\,x + q_{21}\,y - r_{21}\,z + \\ \qquad + 5\left\{\binom{x}{2} p_{22} + x\,y\,s_{211} + \binom{y}{2} q_{22} - \binom{z}{2} r_{22}\right\} + \cdots, \\ F_3 = p_{31}\,x + q_{31}\,y - r_{31}\,z + \\ \qquad + 5\left\{\binom{x}{2} p_{32} + x\,y\,s_{311} + \binom{y}{2} q_{32} - \binom{z}{2} r_{32}\right\} + \cdots. \end{cases}$$

15. Let us first suppose that

$$\Delta_1 = a\,g - b\,f \not\equiv 0, \quad \Delta_2 = f\,d - g\,c \not\equiv 0, \quad \Delta_3 = a\,d - b\,c \not\equiv 0 \pmod 5.$$

According to (6) we have

$$\begin{cases} F_1^{(0)} = a^4\,b\,x + c^4\,d\,y - f^4\,g\,z, \\ F_2^{(0)} \quad 2\,a^3\,b^2\,x \cdot 2\,c^3\,d^2\,y - 2\,f^3\,g^2\,z, \\ F_3^{(0)} \quad 2\,a^2\,b^3\,x \cdot 2\,c^2\,d^3\,y \quad 2\,f^2\,g^3\,z. \end{cases}$$

Since

$$\frac{\partial\,(F_1^{(0)},\ F_2^{(0)},\ F_3^{(0)})}{\partial\,(x,\ y,\ z)} = 4\,a^2\,b\,c^2\,d\,f^2\,g\,\Delta_1\,\Delta_2\,\Delta_3 \not\equiv 0 \pmod 5$$

the pentadic system

$$F_1 = 0, \quad F_2 = 0, \quad F_3 = 0 \quad (5)$$

has only a finite number of integral solutions according to Theorem 3.

16. Now let us consider the case in which one of the numbers Δ_1, Δ_2 and Δ_3 is divisible by 5 while the other two are not divisible by 5. It is of no importance which of the numbers we assume is divisible by 5 since the numbers ε_1, ε_2 and ε_3 are permutable.

Let μ be a natural number and let us suppose that

$$\Delta_1 \begin{cases} \equiv 0 \pmod{5^{\mu}} \\ \not\equiv 0 \pmod{5^{\mu+1}} \end{cases} \quad \text{and} \quad \begin{matrix} \Delta_2 \\ \Delta_3 \end{matrix} \Big\} \not\equiv 0 \pmod 5. \tag{7}$$

It follows from Lemma 1 that we may put

$$a = f + a_1 5^{\mu}, \quad b = g + b_1 5^{\mu}, \tag{8}$$

where a_1 and b_1 are integers.

Putting (8) in (6) we get

$$\begin{cases} F_1^{(\mu)} = (x-z) H_1(x, z) + y K_1(x, y) + 4 f^3 \Delta_1 x, \\ F_2^{(\mu)} = (x-z) H_2(x, z) + y K_2(x, y) - 4 f^2 g \Delta_1 x, \\ F_3^{(\mu)} = (x-z) H_3(x, z) + y K_3(x, y) + 4 f g^2 \Delta_1 x, \end{cases} \tag{9}$$

where for $j = 1, 2, 3$

$$H_j(x, z) = \sum_{i=0}^{\mu} 5^i h_{ij}(x, z), \quad K_j(x, y) = \sum_{i=0}^{\mu} 5^i k_{ij}(x, y), \tag{10}$$

and where h_{ij} and k_{ij} are polynomials with coefficients which are rational integers modulo 5. We have

$$h_{0j} = r_{j1}, \quad k_{0j} = q_{j1} \qquad (j = 1, 2, 3). \tag{11}$$

It is a necessary condition, if the congruence system

$$F_1^{(\mu)} \equiv 0, \quad F_2^{(\mu)} \equiv 0, \quad F_3^{(\mu)} \equiv 0 \pmod{5^{\mu+1}}$$

is to be solvable, that the following system is solvable

$$\begin{cases} (x-z) H_1 + y K_1 - 0 \pmod{5^{\mu}}, \\ (x-z) H_2 + y K_2 \equiv 0 \pmod{5^{\mu}}. \end{cases} \tag{12}$$

It follows from (7), (10) and (11) that

$$\begin{vmatrix} H_1 & K_1 \\ H_2 & K_2 \end{vmatrix} \equiv 2 f^3 g c^3 d \Delta_2 \not\equiv 0 \pmod 5.$$

Hence (12) gives

$$x - z \equiv 0 \quad \text{and} \quad y \equiv 0 \pmod{5^{\mu}}. \tag{13}$$

From (9) we get

$$\frac{\partial (F_1^{(\mu)}, F_2^{(\mu)}, F_3^{(\mu)})}{\partial (x, y, z)} = 5(x - z) G_1(x, y, z) \quad 5 y G_2(x, y, z) - 16 f^4 g^2 c^2 d \Delta_1 \Delta_2^2 \pmod{5^{\mu+1}},$$

where

$$G_1(x, y, z) = \sum_{i \geq 0} 5^i g_{i1}(x, y, z), \quad G_2(x, y, z) = \sum_{i \geq 0} 5^i g_{i2}(x, y, z)$$

and where g_{i1} and g_{i2} are polynomials with coefficients which are rational integers modulo 5. By (7) and (13) we get

$$\frac{\partial (F_1^{(\mu)}, F_2^{(\mu)}, F_3^{(\mu)})}{\partial (x, y, z)} \equiv 0 \pmod{5^{\mu+1}}.$$

Hence the pentadic system

$$F_1 = 0, \quad F_2 = 0, \quad F_3 = 0 \quad (5)$$

has only a finite number of integral solutions according to Theorem 3.

17. Let us regard the case where two of the numbers Δ_1, Δ_2 and Δ_3 are divisible by 5. Then the third number is also divisible by 5 since the following identity holds

$$c \Delta_1 + a \Delta_2 - f \Delta_3 = 0. \quad (14)$$

If $\Delta_j \equiv 0 \pmod 5$ for $j = 1, 2, 3$ then $a \equiv c \equiv f \pmod 5$ and $b \equiv d \equiv g \pmod 5$ according to Lemma 1. Hence we may write

$$a = f + 5 r_1, \quad c = f + 5 r_2, \quad b = g + 5 r_3, \quad d = g + 5 r_4,$$

where r_1, r_2, r_3 and r_4 are integers. Putting this in the first relation of (6) we get

$$F_1 = r_{11}(x + y - z) + 5\{\cdots\} + 5^2\{\cdots\} + \cdots = 0 \quad (5).$$

Since $r_{11} = f^4 g \equiv 0 \pmod 5$ then $x + y - z \equiv 0 \pmod 5$ and we may put

$$x + y - z = 5 t, \quad (15)$$

where t is an integer.

17.1. Suppose that $t=0$. Then $z=x+y$ and (4) in no. **13** may be written

$$\left(\frac{\varepsilon_1}{\varepsilon_3}\right)^x = \left(\frac{\varepsilon_3}{\varepsilon_2}\right)^y \qquad (x\,y \neq 0). \tag{16}$$

If (16) holds for one pair of rational integers x and y different from zero then it has an infinity of solutions in rational integers x and y.

Let us first suppose that

$$\Delta_j \begin{cases} \equiv 0 \pmod{5^\mu} \\ \\ \equiv 0 \pmod{5^{\mu+1}} \end{cases} \qquad (j=1,\ 2,\ 3). \tag{17}$$

By Lemma 1 we get

$$a \equiv c \equiv f \pmod{5^\mu}, \quad b \equiv d \equiv g \pmod{5^\mu},$$

and we may put

$$\begin{cases} a = f + a_1 5^\mu, \quad f = c + f_1 5^\mu, \quad a = c + (f_1 + a_1) 5^\mu, \\ \\ b = g + b_1 5^\mu, \quad g = d + g_1 5^\mu, \quad b = d + (g_1 + b_1) 5^\mu, \end{cases} \tag{18}$$

where a_1, b_1, f_1 and g_1 are integers. Hence

$$\Delta_1 = \Delta_1' 5^\mu, \quad \Delta_2 = \Delta_2' 5^\mu,$$

where

$$\Delta_1' = a_1 g - b_1 f \not\equiv 0 \pmod 5, \quad \Delta_2' = f_1 d - g_1 c \not\equiv 0 \pmod 5,$$

and

$$\Delta_3 = (\Delta_1' + \Delta_2') 5^\mu - (a_1 g_1 - b_1 f_1) 5^{2\mu}. \tag{19}$$

From (17) and (19) it follows that

$$\Delta_1' + \Delta_2' \not\equiv 0 \pmod 5. \tag{20}$$

We find that

$$\frac{1}{\varepsilon_3} = f^4 - f^3 g\,\theta + f^2 g^2 \theta^2 - f g^3 \theta^3 + g^4 \theta^4.$$

Using (18) we get

$$\varepsilon_1 = a + b\,\theta = f + g\,\theta + (a_1 + b_1\,\theta) 5^\mu.$$

Hence it follows that

$$\frac{\varepsilon_1}{\varepsilon_3} = f^5 + M g^5 + 5^\mu \varphi = 1 + 5^\mu \varphi, \tag{21}$$

where

$$\varphi = a_1 f^4 + b_1 M g^4 - \Delta_1' (f^3 \theta - f^2 g \theta^2 + f g^2 \theta^3 - g^3 \theta^4).$$

It is easily seen (Cp. the proof of Lemma 1, no. **5**) that

$$a_1 f^4 + b_1 M g^4 \equiv 0 \pmod 5.$$

We get

$$\varphi^2 \equiv \Delta_1'^2 (-4 f^3 g^3 M + 3 f^2 g^4 M \theta + (f^6 - 2 f g^5 M) \theta^2 + \\ + (g^6 M - 2 f^5 g) \theta^3 + 3 f^4 g^2 \theta^4) \pmod 5.$$

Further we get

$$\frac{1}{\varepsilon_2} = c^4 - c^3 d \theta + c^2 d^2 \theta^2 - c d^3 \theta^3 + d^4 \theta^4,$$

$$\varepsilon_3 = f + g \theta = c + d \theta + (f_1 + g_1 \theta) 5^\mu,$$

which gives

$$\frac{\varepsilon_3}{\varepsilon_2} = 1 + 5^\mu \psi, \tag{22}$$

where

$$\psi = f_1 c^4 + g_1 M d^4 - \Delta_2' (c^3 \theta - c^2 d \theta^2 + c d^2 \theta^3 - d^3 \theta^4).$$

Since

$$f_1 c^4 + g_1 M d^4 \equiv 0 \pmod 5$$

we get

$$\psi^2 \equiv \Delta_2'^2 (-4 c^3 d^3 M + 3 c^2 d^4 M \theta + (c^6 - 2 c d^5 M) \theta^2 + \\ + (d^6 M - 2 c^5 d) \theta^3 + 3 c^4 d^2 \theta^4) \pmod 5.$$

By (21) and (22) we get

$$\frac{1}{5^\mu} \left\{ \left(\frac{\varepsilon_1}{\varepsilon_3}\right)^x - \left(\frac{\varepsilon_3}{\varepsilon_2}\right)^y \right\} = \sum_{k_1=1}^{\infty} \left\{ \binom{x}{k_1} \varphi^{k_1} - \binom{y}{k_1} \psi^{k_1} \right\} 5^{(k_1-1)\mu} = \\ \sum_{r=0}^{4} F_r (x, y) \theta^r, \tag{23}$$

where

$$F_r (x, y) = \sum_{i=0}^{\infty} 5^i f_{ir} (x, y)$$

and where f_{ir} are polynomials with coefficients which are rational integers modulo 5. It follows from (16) and (23) that

$$F_r = 0 \quad (5)$$

for $r = 0, 1, 2, 3, 4$.

We get

$$F_2^{(\mu)} = x\,\Delta_1' f^2 g - y\,\Delta_2'\,c^2 d + \\ + 5^\mu \left\{ \binom{x}{2} (f^6 - 2 f g^5 M)\,\Delta_1'^2 - \binom{y}{2} (c^6 - 2 c d^5 M)\,\Delta_2'^2 \right\},$$

$$F_3^{(\mu)} = -x\,\Delta_1' f g^2 + y\,\Delta_2' c d^2 + \\ + 5^\mu \left\{ \binom{x}{2} (g^6 M - 2 f^5 g)\,\Delta_1'^2 - \binom{y}{2} (d^6 M - 2 c^5 d)\,\Delta_2'^2 \right\}.$$

By (18) we get

$$F_2^{(\mu)} = c^2 d\,(\Delta_1' x - \Delta_2' y) + \\ + 5^\mu \left\{ \binom{x}{2} \Delta_1'^2 - \binom{y}{2} \Delta_2'^2 \right\} (c^6 - 2 c d^5 M) + x c\,\Delta_1' 5^\mu (2\,\Delta_2' + 3 g_1 c),$$

$$F_3^{(\mu)} = -c d^2\,(\Delta_1' x - \Delta_2' y) + \\ + 5^\mu \left\{ \binom{x}{2} \Delta_1'^2 - \binom{y}{2} \Delta_2'^2 \right\} (d^6 M - 2 c^5 d) + x d\,\Delta_1' 5^\mu (2\,\Delta_2' - 3 f_1 d).$$

It is a necessary condition for the congruence system $F_2^{(\mu)} \equiv 0$, $F_3^{(\mu)} \equiv 0 \pmod{5^{\mu+1}}$ to be solvable that

$$\Delta_1' x - \Delta_2' y \equiv 0 \pmod{5^\mu}. \tag{24}$$

If

$$\frac{\partial (F_2^{(\mu)}, F_3^{(\mu)})}{\partial (x, y)} \equiv 0 \pmod{5^{\mu+1}}$$

then, according to (24),

$$5^\mu (\Delta_1' + \Delta_2') \equiv 0 \pmod{5^{\mu+1}},$$

but this is impossible according to (20). Hence the pentadic system

$$F_2 = 0, \quad F_3 = 0 \tag{5}$$

has only a finite number of integral solutions according to Theorem 2. Thus (16) cannot hold if (17) holds.

Now let us suppose that Δ_1, Δ_2 and Δ_3 are divisible by different powers of 5. Without loss of generality we may suppose that Δ_2 is divisible by a higher power of 5 than Δ_1 and Δ_3. It follows from (14) that Δ_1 and Δ_3 are divisible by the same power of 5.

Thus let ϱ and μ be natural numbers and suppose that

$$\Delta_2 \begin{cases} \equiv 0 \pmod{5^\varrho} \\ \not\equiv 0 \pmod{5^{\varrho+1}}, \end{cases} \quad \Delta_1 \Big| \begin{cases} \equiv 0 \pmod{5^\mu} \\ \not\equiv 0 \pmod{5^{\mu+1}}, \end{cases} \quad \Delta_3 \Big| \qquad \varrho > \mu > 0.$$

From Lemma 1 we get

$$f \equiv c,\ g \equiv d \pmod{5^\varrho},\quad a \equiv f,\ b \equiv g \pmod{5^\mu}.$$

Hence we may put

$$f = c + f_1 5^\varrho,\quad g = d + g_1 5^\varrho,\quad a = f + a_1 5^\mu,\quad b = g + b_1 5^\mu, \tag{25}$$

where f_1, g_1, a_1 and b_1 are integers. We have $\Delta_2 = \Delta_2' 5^\varrho$, $\Delta_1 = \Delta_1' 5^\mu$ where

$$\Delta_2' = f_1 d - g_1 c \not\equiv 0 \pmod 5,\quad \Delta_1' = a_1 g - b_1 f \not\equiv 0 \pmod 5.$$

As before we get

$$\frac{\varepsilon_1}{\varepsilon_3} = 1 + 5^\mu \varphi,\quad \frac{\varepsilon_3}{\varepsilon_2} = 1 + 5^\varrho \psi, \tag{26}$$

where

$$\varphi = a_1 f^4 + b_1 M g^4 - \Delta_1' (f^3 \theta - f^2 g \theta^2 + f g^2 \theta^3 - g^3 \theta^4),$$

$$\psi = f_1 c^4 + g_1 M d^4 - \Delta_2' (c^3 \theta - c^2 d \theta^2 + c d^2 \theta^3 - d^3 \theta^4).$$

By $\bar{\mu}_{rn}$ we denote the coefficient of θ^r $(r = 0, 1, 2, 3, 4)$ in φ^n.

It follows from (26) that

$$\frac{1}{5^\mu}\left\{\left(\frac{\varepsilon_1}{\varepsilon_3}\right)^x - \left(\frac{\varepsilon_3}{\varepsilon_2}\right)^y\right\} = x\varphi + \binom{x}{2}\varphi^2 5^\mu + \cdots + \binom{x}{m}\varphi^m 5^{(m-1)\mu} - y\psi 5^{\varrho-\mu} + \cdots = \sum_{r=0}^{4} F_r(x, y)\cdot\theta^r, \tag{27}$$

where $m = \left[\frac{\varrho}{\mu}\right]$ and where

$$F_r(x, y) = \sum_{i=0}^{\infty} 5^i f_{ir}(x, y),$$

where f_{ir} are polynomials with coefficients which are rational integers modulo 5. From (16) and (27) we get

$$F_r = 0 \quad (5)$$

for $r = 0, 1, 2, 3, 4$.

We have

$$\begin{cases} F_2 = x\,\Delta_1' f^2 g + \binom{x}{2} \bar{p}_{22}\, 5^{\mu} + \cdots + \binom{x}{m} \bar{p}_{2m}\, 5^{(m-1)\mu} - y\,\Delta_2' c^2 d\, 5^{\varrho-\mu} + \cdots, \\ F_3 = -x\,\Delta_1' f g^2 + \binom{x}{2} \bar{p}_{32}\, 5^{\mu} + \cdots + \binom{x}{m} \bar{p}_{3m}\, 5^{(m-1)\mu} + y\,\Delta_2' c d^2\, 5^{\varrho-\mu} + \cdots. \end{cases}$$

By (25) we get

$$(28) \quad \begin{cases} F_2^{(\varrho)} = x\,\Delta_1' c^2 d + \binom{x}{2} (c^6 - 2\,c\,d^5\,M)\,\Delta_1'^2\, 5^{\mu} + \cdots - y\,\Delta_2' c^2 d\, 5^{\varrho-\mu} + \cdots, \\ F_3^{(\varrho)} = -x\,\Delta_1' c d^2 + \binom{x}{2} (d^6\,M - 2\,c^5 d)\,\Delta_1'^2\, 5^{\mu} + \cdots + y\,\Delta_2' c d^2\, 5^{\varrho-\mu} + \cdots. \end{cases}$$

$F_2^{(\varrho)}$, $F_3^{(\varrho)}$ each contain only one y-term, since the next y-term in F_2 and F_3 is divisible by $5^{2\varrho-\mu}$. However, $2\varrho-\mu>\varrho$ when $\varrho>\mu$. (28) gives

$$\frac{\partial (F_2^{(\varrho)}, F_3^{(\varrho)})}{\partial (x, y)} \equiv -\frac{2x-1}{2} c^2 d^2\, \Delta_1'^2\, \Delta_2' 5^{\varrho} \pmod{5^{\varrho+1}}.$$

From (28) we get $x \equiv 0 \pmod 5$. Since $c^2 d^2\, \Delta_1'^2\, \Delta_2' \not\equiv 0 \pmod 5$ the pentadic system

$$F_2 = 0, \quad F_3 = 0 \quad (5)$$

has only a finite number of integral solutions according to Theorem 2. Hence it follows that (16) is impossible and we can assume that t in (15) is different from zero.

17.2. Let t in (15) be $\neq 0$ and let η_1 and η_2 be a system of fundamental units in $\boldsymbol{R}(\theta)$. Clearly we may assume that $\eta_1>0$ and $\eta_2>0$. Thus

$$(29) \qquad \varepsilon_1 = \eta_1^{\alpha_1} \eta_2^{\alpha_2}, \quad \varepsilon_2 = \eta_1^{\beta_1} \eta_2^{\beta_2}, \quad \varepsilon_3 = \eta_1^{\gamma_1} \eta_2^{\gamma_2},$$

where α_1, α_2, β_1, β_2, γ_1 and γ_2 are rational integers. At least one of the numbers α_1 and α_2 is different from zero. Similarly, at least one of the numbers β_1, β_2 and at least one of the numbers γ_1, γ_2 is different from zero.

By (4) we get

$$\varepsilon_1^x \varepsilon_2^y \varepsilon_3^{-z} = \eta_1^{x\alpha_1 + y\beta_1 - z\gamma_1} \cdot \eta_2^{x\alpha_2 + y\beta_2 - z\gamma_2} = 1.$$

Since η_1 and η_2 form a system of independent units of $\boldsymbol{R}(\theta)$ we have

$$\begin{cases} x\alpha_1 + y\beta_1 - z\gamma_1 = 0, \\ x\alpha_2 + y\beta_2 - z\gamma_2 = 0, \end{cases} \tag{30}$$

or

$$\begin{cases} (\alpha_1\beta_2 - \beta_1\alpha_2)\, x = z(\gamma_1\beta_2 - \beta_1\gamma_2), \\ (\alpha_1\beta_2 - \beta_1\alpha_2)\, y = z(\gamma_1\alpha_2 - \alpha_1\gamma_2). \end{cases} \tag{31}$$

If one of the numbers

$$\delta_1 = \alpha_1\beta_2 - \beta_1\alpha_2, \quad \delta_2 = \gamma_1\beta_2 - \beta_1\gamma_2, \quad \delta_3 = \gamma_1\alpha_2 - \alpha_1\gamma_2$$

is equal to zero then it follows from (31) that the other two are also equal to zero, since $xyz \neq 0$.

Let us suppose that $\delta_1 = \delta_2 = \delta_3 = 0$. Then all the numbers α_1, α_2, β_1, β_2, γ_1 and $\gamma_2 \neq 0$ because if we had for instance $\alpha_1 = 0$ then α_2 would not be equal to zero. From $\delta_1 = 0$ would follow $\beta_1 = 0$ and thus $\beta_2 \neq 0$. According to (29) we would then have $\varepsilon_1 = \eta_2^{\alpha_2}$ and $\varepsilon_2 = \eta_2^{\beta_2}$ and hence $\varepsilon_1^{\beta_2} = \varepsilon_2^{\alpha_2}$, but this is impossible according to Theorem 4. Hence $\delta_1 = 0$ gives

$$\frac{\alpha_1}{\beta_1} = \frac{\alpha_2}{\beta_2} = \frac{p}{q},$$

where p and q are integers $\neq 0$, $(p, q) = 1$. By putting $\alpha_1 = \beta_1 \cdot \frac{p}{q}$ and $\alpha_2 = \beta_2 \cdot \frac{p}{q}$ we get from (29)

$$\varepsilon_1^q = \varepsilon_2^p$$

which is impossible according to Theorem 4. Hence $\delta_1 \neq 0$, $\delta_2 \neq 0$ and $\delta_3 \neq 0$.

(15) and (30) gives the system

$$\begin{cases} x + y - z = 5t, \\ \alpha_1 x + \beta_1 y - \gamma_1 z = 0, \\ \alpha_2 x + \beta_2 y - \gamma_2 z = 0. \end{cases} \tag{32}$$

From (32) we get

$$x_1 = x\,\bar{\Delta} = 5\delta_2 t, \quad y_1 = y\,\bar{\Delta} = -5\delta_3 t, \quad z_1 = z\,\bar{\Delta} = 5\delta_1 t. \tag{33}$$

Here

$$\Delta = \begin{vmatrix} 1 & 1 & -1 \\ \alpha_1 & \beta_1 & -\gamma_1 \\ \alpha_2 & \beta_2 & -\gamma_2 \end{vmatrix} \neq 0$$

since δ_1, δ_2, δ_3 and $t \neq 0$. If (4) in no. **14** holds for rational integers x, y and z, $xyz \neq 0$, then

$$\varepsilon_1^{5x_1} \varepsilon_2^{5y_1} = \varepsilon_3^{5z_1} \qquad (x_1 y_1 z_1 \neq 0) \tag{34}$$

holds for rational integers x_1, y_1 and z_1 and has an infinite number of solutions.

From (34) we get the following pentadic relation analogous to (6) in no. **14**

$$\bar{F}_1(x_1, y_1, z_1) = p_{11} x_1 + q_{11} y_1 - r_{11} z_1 + \\ + 5\left\{ \binom{x_1}{2} p_{12} + x_1 y_1 s_{111} + \binom{y_1}{2} q_{12} - \binom{z_1}{2} r_{12} \right\} + \cdots = 0 \quad (5), \tag{35}$$

where p_{11}, q_{11}, r_{11}, p_{12} etc. denote the same quantities as in (6). Putting (33) in (35) we get an expression which is a function only of t. If 5^σ is the highest power of 5 such that all the numbers δ_1, δ_2 and δ_3 are divisible by it, it follows from (35) that we can write

$$G(t) = \frac{\bar{F}_1(x_1, y_1, z_1)}{5^{\sigma+1}} = \sum_{i=0}^{\infty} 5^i g_i(t),$$

where $g_i(t)$ are polynomials with coefficients which are rational integers modulo 5. We have $g_0(t) = a_0 t$, where $a_0 \not\equiv 0 \pmod 5$. According to Theorem 1 the pentadic relation

$$G(t) = 0 \quad (5)$$

has only a finite number of integral solutions t. Hence (34) has only a finite number of solutions in rational integers x_1, y_1 and z_1, which implies that relation (34) is impossible.

It follows from this paragraph that the Diophantine equation

$$u^5 + M v^5 = 1$$

cannot have three integral solutions u_1; v_1, u_2; v_2 and u_3; v_3 such that $u_j v_j \not\equiv 0 \pmod 5$ for $j = 1, 2, 3$.

§ 3. $u^5 - Mv^5 = 1$ cannot have three integral solutions u_1; v_1, u_2; v_2 and u_3; v_3 such that $u_1 u_2 u_3 \not\equiv 0 \pmod 5$ and $v_1 v_2 v_3 \not\equiv 0 \pmod 5$.

18. Let us consider the case 2° in no. **13** where $\beta = \delta = \lambda = 0$ and one, two or three of the numbers α, γ and $\varkappa \neq 0$.

We distinguish the following four cases: 1) all three numbers α, γ and $\varkappa$ are different, 2) two of the numbers are equal and less than the third number, 3) two of the numbers are equal and greater than the third number, 4) all three numbers are equal and greater than zero.

We have

$$\varepsilon_1 = 5^\alpha a + b\,\theta, \quad \varepsilon_2 = 5^\gamma c + d\,\theta, \quad \varepsilon_3 = 5^\varkappa f + g\,\theta, \tag{1}$$

which gives

$$\varepsilon_1^5 = 1 + 5^{\alpha+1}\xi, \quad \varepsilon_2^5 = 1 + 5^{\gamma+1}\eta, \quad \varepsilon_3^5 = 1 + 5^{\varkappa+1}\zeta, \tag{2}$$

where

$$\xi = 5^{3\alpha} a^4 b\,\theta + 2\cdot 5^{2\alpha} a^3 b^2 \theta^2 + 2\cdot 5^\alpha a^2 b^3 \theta^3 + a b^4 \theta^4, \tag{3}$$

$$\eta = 5^{3\gamma} c^4 d\,\theta + 2\cdot 5^{2\gamma} c^3 d^2 \theta^2 + 2\cdot 5^\gamma c^2 d^3 \theta^3 + c d^4 \theta^4, \tag{4}$$

$$\zeta = 5^{3\varkappa} f^4 g\,\theta + 2\cdot 5^{2\varkappa} f^3 g^2 \theta^2 + 2\cdot 5^\varkappa f^2 g^3 \theta^3 + f g^4 \theta^4. \tag{5}$$

We denote the coefficient of θ^r $(r = 0, 1, 2, 3, 4)$ in ξ^n, η^n, ζ^n and $\xi^m \eta^n$ by p_{rn}, q_{rn}, r_{rn} and s_{rmn} respectively.

From (2) we get

$$\varepsilon_1^{5x}\varepsilon_2^{5y} - \varepsilon_3^{5z} = \sum_{k_1=0}^{\infty}\binom{x}{k_1}\xi^{k_1}5^{k_1(\alpha+1)}\cdot\sum_{k_2=0}^{\infty}\binom{y}{k_2}\eta^{k_2}5^{k_2(\gamma+1)} - \tag{6}$$

$$-\sum_{k_3=0}^{\infty}\binom{z}{k_3}\zeta^{k_3}5^{k_3(\varkappa+1)} = \sum_{r=0}^{4} F_r(x, y, z)\,\theta^r,$$

where

$$F_r(x, y, z) = \sum_{i=0}^{\infty} 5^i f_{ir}(x, y, z)$$

and where f_{ir} are polynomials with coefficients which are rational integers modulo 5. If $\varepsilon_1^{5x}\varepsilon_2^{5y} = \varepsilon_3^{5z}$ then

$$F_r(x, y, z) = 0 \pmod 5 \tag{7}$$

for $r = 0, 1, 2, 3, 4$.

19. We begin by considering the case where all three numbers α, γ and $\varkappa$ are different. Obviously it is sufficient to treat the case $\alpha > \gamma > \varkappa \geqq 0$. We may write (6)

$$
\begin{aligned}
(8) \qquad \sum_{\tau=0}^{4} F_\tau(x, y, z)\, \theta^\tau = 5^{\varkappa+1} \Big[& x\, \xi\, 5^{\alpha-\varkappa} + \binom{x}{2} \xi^2\, 5^{2(\alpha+1)-(\varkappa+1)} + \\
& + \binom{x}{3} \xi^3\, 5^{3(\alpha+1)-(\varkappa+1)} + \cdots + y\, \eta\, 5^{\gamma-\varkappa} + \binom{y}{2} \eta^2\, 5^{2(\gamma+1)-(\varkappa+1)} + \\
& + \binom{y}{3} \eta^3\, 5^{3(\gamma+1)-(\varkappa+1)} + \cdots + x\, y\, \xi\, \eta\, 5^{\alpha+1+\gamma+1-(\varkappa+1)} + \\
& + x \binom{y}{2} \xi\, \eta^2\, 5^{\alpha+1+2(\gamma+1)-(\varkappa+1)} + \binom{x}{2} y\, \xi^2\, \eta\, 5^{2(\alpha+1)+\gamma+1-(\varkappa+1)} + \\
& + \cdots - z\, \zeta - \binom{z}{2} \zeta^2\, 5^{\varkappa+1} - \binom{z}{3} \zeta^3\, 5^{2(\varkappa+1)} - \cdots \Big].
\end{aligned}
$$

We find that $p_{21} = 2\, a^3\, b^2\, 5^{2\alpha}$, $p_{22} = a^3\, b^2\, (5^{5\alpha}\, a^5 + 4\, M\, b^5)\, 5^\alpha$. Analogous expressions hold for q_{21}, q_{22}, r_{21} and r_{22}. Further we find that

$$s_{211} = a^4\, b\, c^4\, d\, 5^{3(\alpha+\gamma)} + 2\, a^2\, b^3\, c\, d^4\, M\, 5^\alpha + 2\, a\, b^4\, c^2\, d^3\, M\, 5^\gamma.$$

Since $\alpha > \gamma$ then s_{211} is divisible by 5^γ.

By (7) and (8) we get

$$
(9) \qquad \begin{cases}
F_{21}(x, y, z) = \dfrac{F_2(x, y, z)}{5^{3\varkappa+1}} = \displaystyle\sum_{i=0}^{\infty} 5^i\, f_{i\,21}(x, y, z) = 0 \quad (5), \\
F_{31}(x, y, z) = \dfrac{F_3(x, y, z)}{5^{2\varkappa+1}} = \displaystyle\sum_{i=0}^{\infty} 5^i\, f_{i\,31}(x, y, z) = 0 \quad (5), \\
F_{41}(x, y, z) = \dfrac{F_4(x, y, z)}{5^{\varkappa+1}} = \displaystyle\sum_{i=0}^{\infty} 5^i\, f_{i\,41}(x, y, z) = 0 \quad (5),
\end{cases}
$$

where $f_{i\,21}$, $f_{i\,31}$ and $f_{i\,41}$ are polynomials with coefficients which are rational integers modulo 5. We may write

$$
(10) \qquad \begin{cases}
F_{21} = 5^{3(\alpha-\varkappa)}\, x\, H_2(x) + 5^{3(\gamma-\varkappa)}\, y\, K_2(y) + \\
\qquad + 5^{\alpha+2\gamma-3\varkappa+1}\, x\, y\, L_2(x, y) - z\, M_2(z) = 0 \quad (5), \\
F_{31} = 5^{2(\alpha-\varkappa)}\, x\, H_3(x) + 5^{2(\gamma-\varkappa)}\, y\, K_3(y) + 5^{\alpha+\gamma-2\varkappa+1} x\, y\, L_3(x, y) - \\
\qquad - z\, M_3(z) = 0 \quad (5), \\
F_{41} = 5^{\alpha-\varkappa}\, x\, H_4(x) + 5^{\gamma-\varkappa}\, y\, K_4(y) + 5^{\alpha+\gamma-\varkappa+1}\, x\, y\, L_4(x, y) - \\
\qquad - z\, M_4(z) = 0 \quad (5).
\end{cases}
$$

Here

$$(11)\qquad \begin{cases} H_j = \sum_{i=0}^{\infty} 5^i h_{ij}(x), \qquad K_j = \sum_{i=0}^{\infty} 5^i k_{ij}(y), \\ \\ L_j = \sum_{i=0}^{\infty} 5^i l_{ij}(x, y), \quad M_j = \sum_{i=0}^{\infty} 5^i m_{ij}(z). \end{cases} \qquad (j = 2, 3, 4)$$

where h_{ij}, k_{ij}, l_{ij} and m_{ij} are polynomials with coefficients which are rational integers modulo 5.

We have

$$h_{04} = a\, b^4, \quad k_{03} = 2\, c^2\, d^3, \quad m_{02} = 2\, f^3\, g^2.$$

For $k = \alpha + 2\gamma - 3\varkappa$ we get from (10)

$$\frac{\partial (F_{21}^{(k)}, F_{31}^{(k)}, F_{41}^{(k)})}{\partial (x, y, z)} \equiv h_{04}\, k_{03}\, m_{02}\, 5^k \not\equiv 0 \pmod{5^{k+1}}.$$

Hence, according to Theorem 3, the pentadic system

$$F_{21} = 0, \quad F_{31} = 0, \quad F_{41} = 0 \quad (5)$$

has only a finite number of integral solutions x, y and z.

20. Now let us regard the case where two of the numbers α, γ and $\varkappa$ are equal and less than the third number. It suffices to treat the case where $\alpha > \gamma = \varkappa \geqq 0$. Thus we must put $\varkappa = \alpha$ in (1), (2), (5), (6), (8) and (9). We may write

$$(12)\qquad F_{21} = 5^{3(\alpha-\gamma)}\, x\, H_2(x) + y \cdot 2\, c^3\, d^2 - z \cdot 2\, f^3\, g^2 +$$

$$+ 5 \left\{ \binom{y}{2} c^3\, d^2\, (5^{5\gamma}\, c^5 + 4\, M\, d^5) - \binom{z}{2} f^3\, g^2\, (5^{5\gamma}\, f^5 + 4\, M\, g^5) \right\} +$$

$$+ 5^2 \left\{ \binom{y}{3} q_{23} - \binom{z}{3} r_{23} \right\} + \cdots + 5^{\alpha-\gamma+1}\, x\, y\, L_2(x, y) = 0 \quad (5),$$

$$(13)\qquad F_{31} = 5^{2(\alpha-\gamma)}\, x\, H_3(x) + y \cdot 2\, c^2\, d^3 - z \cdot 2\, f^2\, g^3 +$$

$$+ 5 \left\{ \binom{y}{2} q_{32} - \binom{z}{2} r_{32} \right\} + \cdots + 5^{\alpha-\gamma+1}\, x\, y\, L_3(x, y) = 0 \quad (5),$$

$$(14)\qquad F_{41} = 5^{\alpha-\gamma}\, x\, H_4(x) + y \cdot c\, d^4 - z \cdot f\, g^4 +$$

$$+ 5^{\gamma+1} \left\{ \binom{y}{2} q_{42} - \binom{z}{2} r_{42} \right\} + \cdots + 5^{\alpha+1}\, x\, y\, L_4(x, y) = 0 \quad (5).$$

20.1. Let $\Delta_2 = f d - g c \not\equiv 0 \pmod 5$. For $k = \alpha - \gamma$ we get from (12)–(14)

$$\frac{\partial (F_{21}^{(k)}, F_{31}^{(k)}, F_{41}^{(k)})}{\partial (x, y, z)} = 4 a b^4 (c d f g)^2 \Delta_2 5^k \not\equiv 0 \pmod{5^{k+1}}.$$

Hence, according to Theorem 3, the pentadic system (12)–(14) has only a finite number of solutions in rational integers x, y and z.

20.2. Let μ be a natural number and let

$$\Delta_2 \begin{cases} \equiv 0 \pmod{5^\mu} \\ \not\equiv 0 \pmod{5^{\mu+1}}. \end{cases}$$

If $\gamma = 0$ it follows from Lemma 1 and if $\gamma > 0$ it follows from Lemma 2 that $f \equiv c$ and $g \equiv d \pmod{5^\mu}$. Thus we may write

$$f = c + f_1 5^\mu, \quad g = d + g_1 5^\mu, \tag{15}$$

where f_1 and g_1 are integers. We get $\Delta_2 = \Delta_2' 5^\mu$, where

$$\Delta_2' = f_1 d - g_1 c \not\equiv 0 \pmod 5.$$

Putting (15) in (12), (13) and (14) we get

$$F_{21} = 5^{3(\alpha-\gamma)} x H_2(x) + (y - z) N_2(y, z) - 5^\mu T_2(z) + \\ + 5^{\alpha-\gamma+1} x y L_2(x, y) = 0 \quad (5), \tag{16}$$

$$F_{31} = 5^{2(\alpha-\gamma)} x H_3(x) + (y - z) N_3(y, z) - 5^\mu T_3(z) + \\ + 5^{\alpha-\gamma+1} x y L_3(x, y) = 0 \quad (5), \tag{17}$$

$$F_{41} = 5^{\alpha-\gamma} x H_4(x) + (y - z) N_4(y, z) - 5^\mu T_4(z) + \\ + 5^{\alpha+1} x y L_4(x, y) = 0 \quad (5). \tag{18}$$

Here H_j and L_j are analogous to the corresponding functions in (11), further $N_j = \sum_{i=0}^{\infty} 5^i n_{ij}(y, z)$ and $T_j = \sum_{i=0}^{\infty} 5^i t_{ij}(z)$ for $j = 2, 3, 4$, where n_{ij} and t_{ij} are polynomials with coefficients which are rational integers modulo 5.

We have

$$h_{02} = 2\,a^3\,b^2, \quad n_{02} = 2\,c^3\,d^2, \quad t_{02} = -4\,c^2\,d\,\Delta_2' z,$$

$$h_{03} = 2\,a^2\,b^3, \quad n_{03} = 2\,c^2\,d^3, \quad t_{03} = 4\,c\,d^2\,\Delta_2' z,$$

$$h_{04} = a\,b^4, \qquad n_{04} = c\,d^4, \qquad t_{04} = -4\,d^3\,\Delta_2' z.$$

Hence

$$H_j \equiv 0, \quad N_j \equiv 0 \pmod 5 \qquad (j = 2, 3, 4). \tag{19}$$

If $\mu \leqq \alpha - \gamma$ it follows from (18) and (19) that $y - z \equiv 0 \pmod{5^\mu}$. If $\mu \geqq \alpha - \gamma + 1$ it follows from (17) that $y - z \equiv 0 \pmod{5^{\alpha-\gamma+1}}$. Thus, according to (18) and (19), we find that $x \equiv 0 \pmod 5$. Putting $x = 5\,x_1$ in (17) and (18) we get

$$F_{31} = 5^{2(\alpha-\gamma)+1}\,x_1\,H_{31} + (y-z)\,N_3 - 5^\mu\,T_3 + 5^{\alpha-\gamma+2}\,x_1\,y\,L_{31} = 0 \quad (5), \tag{20}$$

$$F_{41} = 5^{\alpha-\gamma+1}\,x_1\,H_{41} + (y-z)\,N_4 - 5^\mu\,T_4 + 5^{\alpha+2}\,x_1\,y\,L_{41} = 0 \quad (5), \tag{21}$$

where $H_{31} = H_3(5\,x_1)$, $L_{31} = L_3(5\,x_1, y)$ etc.

If $\mu \geqq \alpha - \gamma + 2$ it follows from (20) that $y - z \equiv 0 \pmod{5^{\alpha-\gamma+2}}$. Thus, according to (21), $x_1 \equiv 0 \pmod 5$ and we may put $x_1 = 5\,x_2$.

If $\mu \geqq \alpha - \gamma + 3$ we can continue in this manner. Evidently, if $\mu = \alpha - \gamma + n$ $(n = 1, 2, 3, \ldots)$ we get $x = 5^n\,x_n$ and (16)–(18) may be written

$$\begin{cases} F_{21} = 5^{3(\alpha-\gamma)+n}\,x_n\,H_{2n} + (y-z)\,N_2 - 5^\mu\,T_2 + 5^{\alpha-\gamma+n+1}\,x_n\,y\,L_{2n} = 0 \quad (5), \\ F_{31} = 5^{2(\alpha-\gamma)+n}\,x_n\,H_{3n} + (y-z)\,N_3 - 5^\mu\,T_3 + 5^{\alpha-\gamma+n+1}\,x_n\,y\,L_{3n} = 0 \quad (5), \\ F_{41} = 5^{\alpha-\gamma+n}\,x_n\,H_{4n} + (y-z)\,N_4 - 5^\mu\,T_4 + 5^{\alpha+n+1}\,x_n\,y\,L_{4n} = 0 \quad (5), \end{cases}$$

where $H_{jn} = H_j(5^n\,x_n)$ and $L_{jn} = L_j(5^n\,x_n, y)$ for $j = 2, 3, 4$. We thus find that

$$y - z \equiv 0 \pmod{5^\mu} \tag{22}$$

even when $\mu > \alpha - \gamma$.

If $\mu \leqq \alpha - \gamma$ we put $k = \alpha - \gamma + \mu$. If $\mu = \alpha - \gamma + n$ we put

$$k = \alpha - \gamma + n + \mu = 2\,\mu.$$

In all cases we get, according to (22),

$$\frac{\partial(F_{21}^{(k)}, F_{31}^{(k)}, F_{41}^{(k)})}{\partial(x_n, y, z)} \equiv -16\,a\,b^4\,c^4\,d^4\,\Delta_2' 5^k \not\equiv 0 \pmod{5^{k+1}}.$$

Here we have to put $x_n = x$ if $\mu \leqq \alpha - \gamma$. By Theorem 3 it follows that the pentadic system

$$F_{21} = 0, \quad F_{31} = 0, \quad F_{41} = 0 \quad (5)$$

has only a finite number of integral solutions.

21. Now let us investigate the case where two of the numbers α, γ and $\varkappa$ are equal and greater than the third number. Obviously it is sufficient to treat the case $\gamma = \varkappa > \alpha \geqq 0$. We put $\varkappa = \gamma$ in (1), (2), (5) and (6). (6) may be written

$$\begin{aligned}(23)\qquad \sum_{\tau=0}^{4} F_\tau(x, y z)\, \theta^\tau = 5^{\alpha+1} \Bigg[& x\xi + \binom{x}{2} \xi^2 5^{\alpha+1} + \binom{x}{3} \xi^3 5^{2(\alpha+1)} + \cdots + \\ & + 5^{\gamma-\alpha} \{ y\eta - z\zeta \} + 5^{2(\gamma+1)-(\alpha+1)} \left\{ \binom{y}{2} \eta^2 - \binom{z}{2} \zeta^2 \right\} + \\ & + 5^{3(\gamma+1)-(\alpha+1)} \left\{ \binom{y}{3} \eta^3 - \binom{z}{3} \zeta^3 \right\} + \cdots + x y \xi \eta\, 5^{\gamma+1} + \\ & + \binom{x}{2} y \xi^2 \eta\, 5^{\gamma+1+\alpha+1} + x \binom{y}{2} \xi \eta^2 5^{2(\gamma+1)} + \cdots \Bigg].\end{aligned}$$

Since $\gamma > \alpha$ then s_{211} is divisible by 5^α. In the same way as before we get from (7) and (23)

$$\left| \begin{aligned} & F_{21}(x, y, z) = \frac{F_2(x, y, z)}{5^{3\alpha+1}} = \sum_{i=0}^{\infty} 5^i f_{i21}(x, y, z) = 0 \quad (5), \\ & F_{31}(x, y, z) = \frac{F_3(x, y, z)}{5^{2\alpha+1}} = \sum_{i=0}^{\infty} 5^i f_{i31}(x, y, z) = 0 \quad (5), \\ & F_{41}(x, y, z) = \frac{F_4(x, y, z)}{5^{\alpha+1}} = \sum_{i=0}^{\infty} 5^i f_{i41}(x, y, z) = 0 \quad (5), \end{aligned} \right.$$

where f_{i21}, f_{i31} and f_{i41} are polynomials with coefficients which are rational integers modulo 5.

We may write

$$(24)\qquad F_{21} = x H_2(x) + 5^{3(\gamma-\alpha)} U_2(y, z) + 5^{\gamma-\alpha+1} x y L_2(x, y) = 0 \quad (5),$$

$$(25)\qquad F_{31} = x H_3(x) + 5^{2(\gamma-\alpha)} U_3(y, z) + 5^{\gamma-\alpha+1} x y L_3(x, y) = 0 \quad (5),$$

$$(26)\qquad F_{41} = x H_4(x) + 5^{\gamma-\alpha} U_4(y, z) + 5^{\gamma+1} x y L_4(x, y) = 0 \quad (5).$$

Here $H_j = \sum_{i=0}^{\infty} 5^i h_{ij}(x)$, $U_j = \sum_{i=0}^{\infty} 5^i u_{ij}(y, z)$ and $L_j = \sum_{i=0}^{\infty} 5^i l_{ij}(x, y)$, where h_{ij}, u_{ij} and l_{ij} are polynomials with coefficients which are rational integers modulo 5. We have

$$h_{02} = 2a^3 b^2, \quad u_{02} = 2c^3 d^2 y - 2f^3 g^2 z,$$

$$h_{03} = 2a^2 b^3, \quad u_{03} = 2c^2 d^3 y - 2f^2 g^3 z,$$

$$h_{04} = a b^4, \quad u_{04} = c d^4 y - f g^4 z.$$

Evidently

$$H_j \equiv 0 \pmod 5 \qquad (j = 2, 3, 4). \tag{27}$$

It follows from (24) and (27) that $x \equiv 0 \pmod{5^{3(\gamma-\alpha)}}$. Hence we may put $x = 5^{3(\gamma-\alpha)} x_1$, where x_1 is an integer. By using this we get from (24), (25) and (26)

$$\begin{cases} F_{22} = x_1 H_{21} + U_2 + 5^{\gamma-\alpha+1} x_1 y L_{21} = 0 \quad (5), \\ F_{32} = 5^{\gamma-\alpha} x_1 H_{31} + U_3 + 5^{2(\gamma-\alpha)+1} x_1 y L_{31} = 0 \quad (5), \\ F_{42} = 5^{2(\gamma-\alpha)} x_1 H_{41} + U_4 + 5^{2(\gamma-\alpha)+\gamma+1} x_1 y L_{41} = 0 \quad (5). \end{cases} \tag{28}$$

Here $H_{j1} = H_j(5^{3(\gamma-\alpha)} x_1)$ and $L_{j1} = L_j(5^{3(\gamma-\alpha)} x_1, y)$ for $j = 2, 3, 4$. The second index of F_{j2} indicates that F_{j1} has been divided by the largest possible power of 5 such that F_{j2} is still a function of the same type as F_{j1}, i.e. $F_{j2} = \sum_{i=0}^{\infty} 5^i f_{ij2}(x, y, z)$, where f_{ij2} are polynomials with coefficients which are rational integers modulo 5.

Evidently, $F_{j2} = 0$ (5) at the same time as $F_{j1} = 0$ (5).

21.1. Let $\Delta_2 = f d - g c \equiv 0 \pmod 5$. From (28) we get

$$\frac{\partial(F_{22}^{(0)}, F_{32}^{(0)}, F_{42}^{(0)})}{\partial(x_1, y, z)} = 4 a^3 b^2 f g^3 c d^3 \Delta_2 \equiv 0 \pmod 5.$$

Hence, by Theorem 3, the pentadic system

$$F_{22} = 0, \quad F_{32} = 0, \quad F_{42} = 0 \quad (5)$$

has only a finite number of integral solutions.

21.2. Let μ be a natural number and let

$$\Delta_2 \begin{cases} \equiv 0 \pmod{5^\mu} \\ \not\equiv 0 \pmod{5^{\mu+1}}. \end{cases}$$

As before we have $f = c + f_1 5^\mu$, $g = d + g_1 5^\mu$, $\Delta_2 = \Delta_2' 5^\mu$, where $\Delta_2' = f_1 d - g_1 c \not\equiv 0 \pmod 5$. Using this we get from (28)

$$F_{22} = x_1 H_{21} + (y-z) N_2 - 5^\mu T_2 + 5^{\gamma-\alpha+1} x_1 y L_{21} = 0 \quad (5),$$

$$F_{32} = 5^{\gamma-\alpha} x_1 H_{31} + (y-z) N_3 - 5^\mu T_3 + 5^{2(\gamma-\alpha)+1} x_1 y L_{31} = 0 \quad (5),$$

$$F_{42} = 5^{2(\gamma-\alpha)} x_1 H_{41} + (y-z) N_4 - 5^\mu T_4 + 5^{2(\gamma-\alpha)+\gamma+1} x_1 y L_{41} = 0 \quad (5).$$

Here $N_j = \sum_{i=0}^{\infty} 5^i n_{ij}(y, z)$ and $T_j = \sum_{i=0}^{\infty} 5^i t_{ij}(z)$, where n_{ij} and t_{ij} are polynomials with coefficients which are rational integers modulo 5. We have

$$n_{02} = 2 c^3 d^2, \quad t_{02} = -4 c^2 d \Delta_2' z,$$

$$n_{03} = 2 c^2 d^3, \quad t_{03} = 4 c d^2 \Delta_2' z,$$

$$n_{04} = c d^4, \quad t_{04} = -4 d^3 \Delta_2' z.$$

Hence $N_j \not\equiv 0 \pmod 5$. Naturally $H_{j1} \not\equiv 0 \pmod 5$ since h_{0j} is independent of x.

If $\mu \leqq \gamma - \alpha$ it follows from (30) that $y - z \equiv 0 \pmod{5^\mu}$. If $\gamma - \alpha < \mu \leqq 2(\gamma - \alpha)$ it follows from (30) that $y - z \equiv 0 \pmod{5^{\gamma-\alpha}}$, and from (29) that $x_1 \equiv 0 \pmod{5^{\gamma-\alpha}}$. Putting $x_1 = 5^{\gamma-\alpha} x_2$ we get from (29)–(31)

$$(32) \quad \begin{cases} F_{22} = 5^{\gamma-\alpha} x_2 H_{22} + (y-z) N_2 - 5^\mu T_2 + 5^{2(\gamma-\alpha)+1} x_2 y L_{22} = 0 \quad (5), \\ F_{32} = 5^{2(\gamma-\alpha)} x_2 H_{32} + (y-z) N_3 - 5^\mu T_3 + 5^{3(\gamma-\alpha)+1} x_2 y L_{32} = 0 \quad (5), \\ F_{42} = 5^{3(\gamma-\alpha)} x_2 H_{42} + (y-z) N_4 - 5^\mu T_4 + \\ \qquad + 5^{3(\gamma-\alpha)+\gamma+1} x_2 y L_{42} = 0 \quad (5), \end{cases}$$

where the notations are analogous to those we have used above. It follows from (32) that $y - z \equiv 0 \pmod{5^\mu}$. If $\mu > 2(\gamma - \alpha)$ we can continue in this manner. When $(n-1)(\gamma-\alpha) < \mu \leqq n(\gamma-\alpha)$, $n = 1, 2, 3, \ldots$, we get $x_1 = 5^{(n-1)(\gamma-\alpha)} x_n$ and

$$(33)\quad \begin{cases} F_{22} = 5^{(n-1)(\gamma-\alpha)}\, x_n\, H_{2n} + (y-z)\, N_2 - 5^{\mu}\, T_2 + \\ \qquad\qquad + 5^{n(\gamma-\alpha)+1}\, x_n\, y\, L_{2n} = 0 \quad (5), \\ F_{32} = 5^{n(\gamma-\alpha)}\, x_n\, H_{3n} + (y-z)\, N_3 - 5^{\mu}\, T_3 + \\ \qquad\qquad + 5^{(n+1)(\gamma-\alpha)+1}\, x_n\, y\, L_{3n} = 0 \quad (5), \\ F_{42} = 5^{(n+1)(\gamma-\alpha)}\, x_n\, H_{4n} + (y-z)\, N_4 - 5^{\mu}\, T_4 + \\ \qquad\qquad + 5^{(n+1)(\gamma-\alpha)+\gamma+1}\, x_n\, y\, L_{4n} = 0 \quad (5). \end{cases}$$

We find that $y - z \equiv 0 \pmod{5^{\mu}}$. For $k = (n-1)(\gamma-\alpha) + \mu$ we get from (33)

$$\frac{\partial\,(F_{22}^{(k)},\ F_{32}^{(k)},\ F_{42}^{(k)})}{\partial\,(x_n,\ y,\ z)} = 24\, a^3\, b^2\, c^2\, d^6\, \Delta_2'\, 5^k \not\equiv 0 \pmod{5^{k+1}}.$$

Hence, according to Theorem 3, the pentadic system

$$F_{22} = 0, \quad F_{32} = 0, \quad F_{42} = 0 \quad (5)$$

has only a finite number of integral solutions.

22. Finally let us assume that $\alpha = \gamma = \varkappa > 0$. We put $\gamma = \alpha$ and $\varkappa = \alpha$ in (1), (2), (4), (5) and (6). (6) may therefore be written thus

$$\sum_{\nu=1}^{\infty} 5^{\nu(\alpha-1)} \left\{ \sum_{s=0}^{\nu} \left[\binom{x}{\nu-s} \binom{y}{s} \xi^{\nu-s}\, \eta^s \right] - \binom{z}{\nu} \zeta^{\nu} \right\} = \sum_{r=0}^{4} F_r\,(x,\ y,\ z)\, 0^r.$$

We write

$$(34)\quad \begin{cases} F_{21}(x,\ y,\ z) = \dfrac{F_2\,(x,\ y,\ z)}{5^{3\alpha+1}} = \sum\limits_{t=0}^{\infty} 5^t\, f_{t21}\,(x,\ y,\ z), \\ F_{31}(x,\ y,\ z) = \dfrac{F_3\,(x,\ y,\ z)}{5^{2\alpha+1}} = \sum\limits_{t=0}^{\infty} 5^t\, f_{t31}\,(x,\ y,\ z), \\ F_{41}(x,\ y,\ z) = \dfrac{F_4\,(x,\ y,\ z)}{5^{\alpha+1}} = \sum\limits_{t=0}^{\infty} 5^t\, f_{t41}\,(x,\ y,\ z), \end{cases}$$

where f_{t21}, f_{t31} and f_{t41} are polynomials with coefficients which are rational integers modulo 5. By (7) and (34) we get

$$(35)\qquad F_{21} = 0, \quad F_{31} = 0, \quad F_{41} = 0 \quad (5).$$

By p'_{rn}, q'_{rn}, etc. we denote that the number p_{rn}, q_{rn}, etc. has been divided by the largest possible power of 5, e.g. $p_{21} = 5^{2a} p'_{21}$, $p_{22} = 5^a p'_{22}$. We may write

$$
\left\{
\begin{aligned}
& F_{21} - 2a^3 b^2 x + 2c^3 d^2 y - 2f^3 g^2 z + \\
& + 5\left\{ \binom{x}{2} p'_{22} + x\, y\, s'_{211} + \binom{y}{2} q'_{22} - \binom{z}{2} r'_{22} \right\} + \\
& + 5^2 \left\{ \binom{x}{3} p_{23} + \binom{x}{2} y\, s_{221} + x \binom{y}{2} s_{212} + \binom{y}{3} q_{23} - \binom{z}{3} r_{23} \right\} + \cdots, \\
& F_{31} = 2a^2 b^3 x + 2c^2 d^3 y - 2f^2 g^3 z + \\
& + 5\left\{ \binom{x}{2} p_{32} + x\, y\, s_{311} + \binom{y}{2} q_{32} - \binom{z}{2} r_{32} \right\} + \cdots, \\
& F_{41} = a b^4 x + c d^4 y - f g^4 z + \\
& + 5^{a+1} \left\{ \binom{x}{2} p_{42} + x\, y\, s_{411} + \binom{y}{2} q_{42} - \binom{z}{2} r_{42} \right\} + \cdots.
\end{aligned}
\right. \tag{36}
$$

22.1. Let us first suppose that

$$\Delta_1 = ag - bf \not\equiv 0, \quad \Delta_2 = fd - gc \not\equiv 0, \quad \Delta_3 = ad - bc \not\equiv 0 \pmod 5.$$

From (36) we then get

$$\frac{\partial (F_{21}^{(0)}, F_{31}^{(0)}, F_{41}^{(0)})}{\partial (x, y, z)} = 4\, a b^2 c d^2 f g^2 \Delta_1 \Delta_2 \Delta_3 \not\equiv 0 \pmod 5.$$

Hence, according to Theorem 3, the pentadic system (35) has only a finite number of integral solutions under the above suppositions.

22.2. Now let us consider the case in which one of the numbers Δ_1, Δ_2 and Δ_3 is divisible by 5 while the other two are not divisible by 5. There is no loss of generality in supposing that

$$\Delta_1 \begin{cases} \equiv 0 \pmod{5^\mu} & \Delta_2 \\ \not\equiv 0 \pmod{5^{\mu+1}}, & \Delta_3 \end{cases} \not\equiv 0 \pmod 5,$$

where μ is a natural number. It follows from Lemma 2 no. **7** that we may write

$$a = f + a_1 5^\mu, \quad b = g + b_1 5^\mu, \tag{37}$$

where a_1 and b_1 are integers.

Putting (37) in (36) we get

$$(38)\qquad \begin{cases} F_{21}^{(\mu)} = (x-z)\,H_2(x,\,z) + y\,K_2(x,\,y) - 4\,f^2 g\,\Delta_1 x, \\ F_{31}^{(\mu)} = (x-z)\,H_3(x,\,z) + y\,K_3(x,\,y) + 4\,f\,g^2\,\Delta_1 x, \\ F_{41}^{(\mu)} = (x-z)\,H_4(x,\,z) + y\,K_4(x,\,y) - 4\,g^3\,\Delta_1 x. \end{cases}$$

For $j = 2, 3, 4$ we have

$$H_j(x,\,z) = \sum_{i \geq 0} 5^i\, h_{ij}(x,\,z), \quad K_j(x,\,y) = \sum_{i \geq 0} 5^i\, k_{ij}(x,\,y),$$

where h_{ij} and k_{ij} are polynomials with coefficients which are rational integers modulo 5. We have

$$h_{02} = 2\,f^3 g^2, \quad k_{02} = 2\,c^3 d^2,$$

$$h_{03} = 2\,f^2 g^3, \quad k_{03} = 2\,c^2 d^3,$$

$$h_{04} = f\,g^4, \qquad k_{04} = c\,d^4.$$

Since

$$H_2 K_3 - K_2 H_3 \equiv 4\,(c\,d\,f\,g)^2\,\Delta_2 \not\equiv 0 \pmod 5$$

(38) gives

$$(39)\qquad x - z \equiv 0, \quad y \equiv 0 \pmod{5^\mu}.$$

With the aid of (39) we get from (38)

$$\frac{\partial(F_{21}^{(\mu)},\, F_{31}^{(\mu)},\, F_{41}^{(\mu)})}{\partial(x,\, y,\, z)} \equiv -16\,g^4 f^2 d^2 c\,\Delta_1\,\Delta_2^2 \not\equiv 0 \pmod{5^{\mu+1}}.$$

Hence, according to Theorem 3, the pentadic system (35) has only a finite number of integral solutions under the above assumptions.

22.3. Let us regard the case where two of the numbers Δ_1, Δ_2 and Δ_3 are divisible by 5. As before (no. **17**) we find that the third number is also divisible by 5 and that

$$x + y - z = 5\,t,$$

where t is an integer. If $t \neq 0$ we may apply exactly the same reasoning as in no. **17.2.** Thus we confine ourselves to the case $t = 0$. Then $z = x + y$ and (4) in no. **13** may be written

$$(40)\qquad \left(\frac{\varepsilon_1}{\varepsilon_3}\right)^x = \left(\frac{\varepsilon_3}{\varepsilon_2}\right)^y \qquad (x\,y \neq 0).$$

Let us first suppose that Δ_1, Δ_2 and Δ_3 are divisible by the same power of 5:

$$\Delta_j \begin{cases} \equiv 0 \pmod{5^\mu} \\ \not\equiv 0 \pmod{5^{\mu+1}} \end{cases} \qquad (j = 1, 2, 3).$$

According to Lemma 2 we may put

$$(41) \qquad \begin{cases} a = f + a_1 5^\mu, \quad f = c + f_1 5^\mu, \quad a = c + (f_1 + a_1) 5^\mu, \\ b = g + b_1 5^\mu, \quad g = d + g_1 5^\mu, \quad b = d + (g_1 + b_1) 5^\mu. \end{cases}$$

Then $\Delta_1 = \Delta_1' 5^\mu$, $\Delta_2 = \Delta_2' 5^\mu$ and $\Delta_3 = (\Delta_1' + \Delta_2') 5^\mu - (a_1 g_1 - b_1 f_1) 5^{2\mu}$, where

$$(42) \qquad \Delta_1' = a_1 g - b_1 f \not\equiv 0, \quad \Delta_2' = f_1 d - g_1 c \not\equiv 0, \quad \Delta_1' + \Delta_2' \not\equiv 0 \pmod 5.$$

We find that

$$\frac{1}{\varepsilon_3} = 5^{4\alpha} f^4 - 5^{3\alpha} f^3 g\, \theta + 5^{2\alpha} f^2 g^2 \theta^2 - 5^\alpha f g^3 \theta^3 + g^4 \theta^4,$$

$$\varepsilon_1 = 5^\alpha a + b\, \theta = 5^\alpha f + g\, \theta + (5^\alpha a_1 + b_1 \theta) 5^\mu,$$

which gives

$$(43) \qquad \frac{\varepsilon_1}{\varepsilon_3} = 1 + 5^\mu \varphi,$$

where

$$\varphi = 5^{5\alpha} a_1 f^4 + b_1 M g^4 - \Delta_1' (5^{4\alpha} f^3 \theta - 5^{3\alpha} f^2 g\, \theta^2 + 5^{2\alpha} f g^2 \theta^3 - 5^\alpha g^3 \theta^4).$$

Further we find that

$$\frac{1}{\varepsilon_2} = 5^{4\alpha} c^4 - 5^{3\alpha} c^3 d\, \theta + 5^{2\alpha} c^2 d^2 \theta^2 - 5^\alpha c d^3 \theta^3 + d^4 \theta^4,$$

$$\varepsilon_3 = 5^\alpha f + g\, \theta = 5^\alpha c + d\, \theta + (5^\alpha f_1 + g_1 \theta) 5^\mu,$$

which gives

$$(44) \qquad \frac{\varepsilon_3}{\varepsilon_2} = 1 + 5^\mu \psi,$$

where

$$\psi = 5^{5\alpha} f_1 c^4 + g_1 M d^4 - \Delta_2' (5^{4\alpha} c^3 \theta - 5^{3\alpha} c^2 d \theta^2 + 5^{2\alpha} c d^2 \theta^3 - 5^{\alpha} d^3 \theta^4).$$

It follows from (43) and (44) that

$$\frac{1}{5^{\mu}} \left\{ \left(\frac{\varepsilon_1}{\varepsilon_3}\right)^x - \left(\frac{\varepsilon_3}{\varepsilon_2}\right)^y \right\} = x\varphi - y\psi + 5^{\mu} \left\{ \binom{x}{2} \varphi^2 - \binom{y}{2} \psi^2 \right\} + \cdots = \sum_{r=0}^{4} F_r(x, y) \cdot \theta^r. \tag{45}$$

Here

$$F_r(x, y) = \sum_{t=0}^{\infty} 5^t f_{tr}(x, y),$$

where f_{tr} are polynomials with coefficients which are rational integers modulo 5. It follows from (40) and (45) that

$$F_r = 0 \quad (5) \tag{46}$$

for $r = 0, 1, 2, 3, 4$.

Both φ and ψ are numbers in $\boldsymbol{R}(\theta)$ of the form

$$\chi = A_0 + 5^{4\alpha} A_1 \theta + 5^{3\alpha} A_2 \theta^2 + 5^{2\alpha} A_3 \theta^3 + 5^{\alpha} A_4 \theta^4.$$

It is easily seen that any power of χ, say χ^h, (h natural number) may be written

$$\chi^h = B_0 + 5^{4\alpha} B_1 \theta + 5^{3\alpha} B_2 \theta^2 + 5^{2\alpha} B_3 \theta^3 + 5^{\alpha} B_4 \theta^4,$$

where the numbers B_i are rational integers. Therefore it follows from (45) that we may write

$$\begin{cases} F_{21}(x, y) = \dfrac{F_2(x, y)}{5^{3\alpha}} = \sum_{t=0}^{\infty} 5^t f_{t21}(x, y), \\ F_{31}(x, y) = \dfrac{F_3(x, y)}{5^{2\alpha}} = \sum_{t=0}^{\infty} 5^t f_{t31}(x, y), \end{cases} \tag{47}$$

where f_{t21} and $f^t{}_{31}$ are polynomials with coefficients which are rationa integers modulo 5. By (46) and (47) we get

$$F_{21} = 0, \quad F_{31} = 0 \quad (5). \tag{48}$$

We have

$$\left\{\begin{array}{l} F_{21}^{(\mu)} = x \cdot \Delta_1' f^2 g - y \cdot \Delta_2' c^2 d - 5^\mu \left\{ \binom{x}{2} 2 f g^5 M \Delta_1'^2 - \binom{y}{2} 2 c d^5 M \Delta_2'^2 \right\}, \\ F_{31}^{(\mu)} = -x \cdot \Delta_1' f g^2 + y \cdot \Delta_2' c d^2 + 5^\mu \left\{ \binom{x}{2} g^6 M \Delta_1'^2 - \binom{y}{2} d^6 M \Delta_2'^2 \right\}. \end{array}\right.$$

By (41) we get

$$\left\{\begin{array}{l} F_{21}^{(\mu)} = c^2 d (\Delta_1' x - \Delta_2' y) - 5^\mu \left\{ \binom{x}{2} \Delta_1'^2 - \binom{y}{2} \Delta_2'^2 \right\} 2 M c d^5 + \\ \qquad + x \cdot c \Delta_1' 5^\mu (2 \Delta_2' + 3 g_1 c), \\ F_{31}^{(\mu)} = c d^2 (\Delta_1' x - \Delta_2' y) + 5^\mu \left\{ \binom{x}{2} \Delta_1'^2 - \binom{y}{2} \Delta_2'^2 \right\} M d^6 + \\ \qquad + x d \Delta_1' 5^\mu (2 \Delta_2' - 3 f_1 d). \end{array}\right.$$

Since $\Delta_1' x - \Delta_2' y \equiv 0 \pmod{5^\mu}$ it follows from the congruence

$$\frac{\partial (F_{21}^{(\mu)}, F_{31}^{(\mu)})}{\partial (x, y)} \equiv 0 \pmod{5^{\mu+1}}$$

that $\Delta_1' + \Delta_2' \equiv 0 \pmod 5$, but this is impossible according to (42). Hence, by Theorem 2, the pentadic system (48) has only a finite number of integral solutions when Δ_1, Δ_2 and Δ_3 are all divisible by the same power of 5. Thus equation (40) is impossible in this case.

Now let us suppose that Δ_1, Δ_2 and Δ_3 are divisible by different powers of 5. We may suppose that Δ_2 is divisible by a higher power of 5 than Δ_1 and Δ_3. As before we find that Δ_1 and Δ_3 are divisible by the same power of 5. In fact, we may suppose that

$$\Delta_2 \left\{\begin{array}{l} \equiv 0 \pmod{5^\varrho} \\ \not\equiv 0 \pmod{5^{\varrho+1}}, \end{array}\right. \qquad \left.\begin{array}{l} \Delta_1 \\ \Delta_3 \end{array}\right\} \left\{\begin{array}{l} \equiv 0 \pmod{5^\mu} \\ \not\equiv 0 \pmod{5^{\mu+1}}, \end{array}\right.$$

where $\varrho > \mu > 0$. It follows from Lemma 2 that we may put

$$(49) \qquad f = c + f_1 5^\varrho, \quad g = d + g_1 5^\varrho, \quad a = f + a_1 5^\mu, \quad b = g + b_1 5^\mu.$$

We have $\Delta_2 = \Delta_2' 5^\varrho$ and $\Delta_1 = \Delta_1' 5^\mu$, where

$$\Delta_2' = f_1 d - g_1 c \not\equiv 0 \pmod 5, \quad \Delta_1' = a_1 g - b_1 f \not\equiv 0 \pmod 5.$$

As before we get

$$\frac{\varepsilon_1}{\varepsilon_3} = 1 + 5^{\mu}\varphi, \quad \frac{\varepsilon_3}{\varepsilon_2} = 1 + 5^{\varrho}\psi, \tag{50}$$

where

$$\varphi = 5^{5\alpha} a_1 f^4 + b_1 M g^4 - \Delta_1' (5^{4\alpha} f^3 \theta - 5^{3\alpha} f^2 g \theta^2 + 5^{2\alpha} f g^2 \theta^3 - 5^{\alpha} g^3 \theta^4),$$

$$\psi = 5^{5\alpha} f_1 c^4 + g_1 M d^4 - \Delta_2' (5^{4\alpha} c^3 \theta - 5^{3\alpha} c^2 d \theta^2 + 5^{2\alpha} c d^2 \theta^3 - 5^{\alpha} d^3 \theta^4).$$

From (50) we get

$$\frac{1}{5^{\mu}} \left\{ \left(\frac{\varepsilon_1}{\varepsilon_3}\right)^x - \left(\frac{\varepsilon_3}{\varepsilon_2}\right)^y \right\} = x\varphi + \binom{x}{2} \varphi^2 5^{\mu} + \cdots + \binom{x}{m} \varphi^m 5^{(m-1)\mu} - y\psi 5^{\varrho-\mu} + \cdots \sum_{r=0}^{4} F_r(x, y)\theta^r, \tag{51}$$

where $m = \left[\frac{\varrho}{\mu}\right]$, and where

$$F_r(x, y) = \sum_{i=0}^{\infty} 5^i f_{ir}(x, y),$$

where f_{ir} are polynomials with coefficients which are rational integers modulo 5. By (40) and (51) we get

$$F_r = 0 \quad (5) \tag{52}$$

for $r = 0, 1, 2, 3, 4$. We find that

$$\left| \begin{array}{l} F_{21}(x, y) = \dfrac{F_2(x, y)}{5^{3\alpha}} = \sum\limits_{i=0}^{\infty} 5^i f_{i21}(x, y), \\ \\ F_{31}(x, y) = \dfrac{F_3(x, y)}{5^{2\alpha}} = \sum\limits_{i=0}^{\infty} 5^i f_{i31}(x, y), \end{array} \right.$$

where f_{i21} and f_{i31} are polynomials of the same kind as f_{ir} for $0 \leqq r \leqq 4$. It follows from (52) that

$$F_{21} = 0, \quad F_{31} = 0 \quad (5). \tag{53}$$

We have

$$\left| \begin{array}{l} F_{21} = x \Delta_1' f^2 g - \binom{x}{2} 2 f g^5 M \Delta_1'^2 5^{\mu} + \cdots - y \Delta_2' c^2 d 5^{\varrho-\mu} + \cdots, \\ \\ F_{31} = x \Delta_1' f g^2 + \binom{x}{2} g^6 M \Delta_1'^2 5^{\mu} + \cdots + y \Delta_2' c d^2 5^{\varrho-\mu} + \cdots. \end{array} \right.$$

By (49) we get

$$(54)\quad \begin{cases} F_{21}^{(\varrho)} = x\,\Delta_1'\,c^2\,d - \binom{x}{2} 2\,c\,d^5\,M\,\Delta_1'^2\,5^{\mu} + \cdots - y\,\Delta_2'\,c^2\,d\,5^{\varrho-\mu} + \cdots, \\ F_{31}^{(\varrho)} = -x\,\Delta_1'\,c\,d^2 + \binom{x}{2} d^6\,M\,\Delta_1'^2\,5^{\mu} + \cdots + y\,\Delta_2'\,c\,d^2\,5^{\varrho-\mu} + \cdots. \end{cases}$$

Since $x \equiv 0 \pmod 5$ and $M \not\equiv 0 \pmod 5$, it follows from (54) that

$$\frac{\partial\,(F_{21}^{(\varrho)},\, F_{31}^{(\varrho)})}{\partial\,(x,\, y)} \equiv c^2\,d^7\,M\,\Delta_1'^2\,\Delta_2'\,5^{\varrho} \not\equiv 0 \pmod{5^{\varrho+1}}.$$

Hence, according to Theorem 2, the pentadic system (54) has only a finite number of integral solutions. Consequently equation (40) is impossible in all cases.

It follows from this paragraph that the equation

$$u^5 + Mv^5 = 1$$

cannot have three integral solutions u_1; v_1, u_2; v_2 and u_3; v_3 such that $u_1 u_2 u_3 \equiv 0 \pmod 5$ and $v_1 v_2 v_3 \not\equiv 0 \pmod 5$.

§ 4. $u^5 + Mv^5 = 1$ cannot have three non-trivial integral solutions u_1; v_1, u_2; v_2 and u_3; v_3 such that $u_1 u_2 u_3 \not\equiv 0 \pmod 5$ and $v_1 v_2 v_3 \equiv 0 \pmod 5$.

23. Let us consider the case 5° in no. **13**, where $\alpha = \gamma = \varkappa = 0$ and one, two or three of the numbers β, δ and $\lambda \neq 0$.

We have

$$(1)\qquad \varepsilon_1 = a + 5^{\beta}\,b\,\theta, \quad \varepsilon_2 = c + 5^{\delta}\,d\,\theta, \quad \varepsilon_3 = f + 5^{\lambda}\,g\,\theta,$$

which gives

$$(2)\qquad \varepsilon_1^5 = 1 + 5^{\beta+1}\,\xi, \quad \varepsilon_2^5 = 1 + 5^{\delta+1}\,\eta, \quad \varepsilon_3^5 = 1 + 5^{\lambda+1}\,\zeta,$$

where

$$(3)\quad \begin{cases} \xi = a^4\,b\,\theta + 2\cdot 5^{\beta}\,a^3\,b^2\,\theta^2 + 2\cdot 5^{2\beta}\,a^2\,b^3\,\theta^3 + 5^{3\beta}\,a\,b^4\,\theta^4, \\ \eta = c^4\,d\,\theta + 2\cdot 5^{\delta}\,c^3\,d^2\,\theta^2 + 2\cdot 5^{2\delta}\,c^2\,d^3\,\theta^3 + 5^{3\delta}\,c\,d^4\,\theta^4, \\ \zeta = f^4\,g\,\theta + 2\cdot 5^{\lambda}\,f^3\,g^2\,\theta^2 + 2\cdot 5^{2\lambda}\,f^2\,g^3\,\theta^3 + 5^{3\lambda}\,f\,g^4\,\theta^4. \end{cases}$$

If we compare these expressions, which are fundamental for the operations, with the corresponding expressions in no. **18**, we find that the above-mentioned formulas are symmetrical in relation to those in no. **18**. It is clear that the operations in this case will be analogous to the operations in § 3, just as the treatment of the cases in no. **11–12** was analogous to the treatment of the symmetrical cases in no. **6–7**. To gain complete symmetry, however, we must, proceeding from the expression

$$\varepsilon_1^{5^x}\,\varepsilon_2^{5^y} - \varepsilon_3^{5^z} = \sum_{r=0}^{4} F_r(x, y, z)\,\theta^r,$$

consider the functions F_1, F_2 and F_3 and not, as in § 3, the functions F_2, F_3 and F_4. When two or three of the numbers β, δ and λ are equal we have to use Lemma 3 (no. **12**) instead of Lemma 2 (no. **7**).

To avoid repetition and because of the length of the proofs we shall not work them out here. However, it is evident that the equation $u^5 + Mv^5 = 1$ cannot have three non-trivial integral solutions u_1; v_1, u_2; v_2 and u_3; v_3 such that $u_1 u_2 u_3 \not\equiv 0 \pmod 5$ and

$$v_1 v_2 v_3 \equiv 0 \pmod 5.$$

§ 5. $u^5 + Mv^5 = 1$ cannot have three non-trivial integral solutions u_1; v_1, u_2; v_2 and u_3; v_3 such that $v_1 u_2 u_3 \equiv 0 \pmod 5$ and $u_1 v_2 v_3 \not\equiv 0 \pmod 5$.

24. Let us consider the case in which the coefficient of θ in one of the numbers ε_1, ε_2 and ε_3 is divisible by 5, while the coefficient of θ in the other two numbers is not divisible by 5. Clearly it is sufficient to treat the case in which

$$(1) \qquad \varepsilon_1 = 5^\alpha a + b\,\theta, \quad \varepsilon_2 = c + 5^\delta d\,\theta, \quad \varepsilon_3 = 5^\varkappa f + g\,\theta,$$

where α and $\varkappa$ are $\geqq 0$ and $\delta > 0$, i.e. the case 3° in no. **13.**

We must examine the following cases:

1) $\alpha \geqq \varkappa \geqq \delta > 0$, 2) $\alpha \geqq \delta \geqq \varkappa \geqq 0$, 3) $\delta \geqq \alpha \geqq \varkappa \geqq 0$.

Here we distinguish the following possibilities:

1 a) all three numbers α, $\varkappa$, δ are different:

$$\alpha > \varkappa > \delta > 0,$$

1 b) two of the numbers are equal and greater than the third number:

$$\alpha = \varkappa > \delta > 0,$$

1 c) two of the numbers are equal and less than the third number:

$$\alpha > \varkappa = \delta > 0,$$

1 d) all three numbers are equal and greater than zero:

$$\alpha = \varkappa = \delta > 0;$$

2 a) $\alpha > \delta > \varkappa \geqq 0$,

2 b) $\alpha = \delta > \varkappa \geqq 0$,

2 c) $\alpha > \delta = \varkappa > 0$, this case coincides with case 1 c),

2 d) $\alpha = \delta = \varkappa > 0$, this case coincides with case 1 d);

3 a) $\delta > \alpha > \varkappa \geqq 0$,

3 b) $\delta = \alpha > \varkappa \geqq 0$, this case coincides with case 2 b),

3 c) $\delta > \alpha = \varkappa \geqq 0$,

3 d) $\delta = \alpha = \varkappa \geqq 0$, this case coincides with case 1 d).

By (1) we get

$$\varepsilon_1^5 = 1 + 5^{\alpha+1}\xi, \quad \varepsilon_2^5 = 1 + 5^{\delta+1}\eta, \quad \varepsilon_3^5 = 1 + 5^{\varkappa+1}\zeta, \tag{2}$$

where

$$\begin{cases} \xi = 5^{3\alpha} a^4 b\, \theta + 2 \cdot 5^{2\alpha} a^3 b^2 \theta^2 + 2 \cdot 5^{\alpha} a^2 b^3 \theta^3 + a\, b^4 \theta^4, \\ \eta = c^4 d\, \theta + 2 \cdot 5^{\delta} c^3 d^2 \theta^2 + 2 \cdot 5^{2\delta} c^2 d^3 \theta^3 + 5^{3\delta} c\, d^4 \theta^4, \\ \zeta = 5^{3\varkappa} f^4 g\, \theta + 2 \cdot 5^{2\varkappa} f^3 g^2 \theta^2 + 2 \cdot 5^{\varkappa} f^2 g^3 \theta^3 + f\, g^4 \theta^4. \end{cases} \tag{3}$$

It follows from (2) that

$$\varepsilon_1^{5x}\, \varepsilon_2^{5y} - \varepsilon_3^{5z} = x\, \xi\, 5^{\alpha+1} + \binom{x}{2} \xi^2 5^{2(\alpha+1)} + \binom{x}{3} \xi^3 5^{3(\alpha+1)} + \cdots + \tag{4}$$

$$+ y\, \eta\, 5^{\delta+1} + \binom{y}{2} \eta^2 5^{2(\delta+1)} + \binom{y}{3} \eta^3 5^{3(\delta+1)} + \cdots + x\, y\, \xi\, \eta\, 5^{\alpha+1+\delta+1} +$$

$$+ x \binom{y}{2} \xi\, \eta^2 5^{\alpha+1+2(\delta+1)} + x \binom{y}{3} \xi\, \eta^3 5^{\alpha+1+3(\delta+1)} + x \binom{y}{4} \xi\, \eta^4 5^{\alpha+1+4(\delta+1)} +$$

$$+ \cdots + \binom{x}{2} y\, \xi^2 \eta\, 5^{2(\alpha+1)+\delta+1} + \binom{x}{2}\binom{y}{2} \xi^2 \eta^2\, 5^{2(\alpha+1)+2(\delta+1)} +$$

$$+ \binom{x}{2}\binom{y}{3} \xi^2 \eta^3\, 5^{2(\alpha+1)+3(\delta+1)} + \cdots + \binom{x}{3} y\, \xi^3 \eta\, 5^{3(\alpha+1)+\delta+1} +$$

$$+ \binom{x}{3}\binom{y}{2} \xi^3 \eta^2\, 5^{3(\alpha+1)+2(\delta+1)} + \cdots - z\, \zeta\, 5^{\varkappa+1} - \binom{z}{2} \zeta^2\, 5^{2(\varkappa+1)} -$$

$$\binom{z}{3} \zeta^3\, 5^{3(\varkappa+1)} - \cdots = \sum_{r=0}^{4} F_r(x, y, z)\, \theta^r,$$

where

$$F_r(x, y, z) = \sum_{i=0}^{\infty} 5^i f_{ir}(x, y, z),$$

and where f_{ir} are polynomials with coefficients which are rational integers modulo 5. If $\varepsilon_1^{5x}\, \varepsilon_2^{5y} = \varepsilon_3^{5z}$ it follows from (4) that

$$F_r = 0 \quad (5)$$

for $r = 0, 1, 2, 3, 4$.

As before we denote by p_{rn}, q_{rn}, r_{rn} and s_{rmn} the coefficient of θ^r $(r = 0, 1, 2, 3, 4)$ in ξ^n, η^n, ζ^n and $\xi^m \eta^n$ respectively. By p'_{rn}, q'_{rn} etc. we denote, where necessary, p_{rn}, q_{rn} etc. divided by the largest possible power of 5.

25. We begin by considering the case $\alpha > \varkappa > \delta > 0$. In order to clarify the subsequent operations we give a table below. It shows the exponents of the highest power of 5 by which the coefficients of θ^r $(r = 0, 1, 2, 3, 4)$ in the respective expressions are divisible. In calculating the lower part of the table we have assumed that $\alpha \geqq \delta$. The values in round brackets must be used where $3\delta \geqq 2\alpha$, the values in square brackets must be used where $4\delta \geqq \alpha$.

We may write

$$F_{11}(x, y, z) = \frac{F_1(x, y, z)}{5^{\delta+1}} = \sum_{i=0}^{\infty} 5^i f_{i11}(x, y, z),$$

$$F_{21}(x, y, z) = \frac{F_2(x, y, z)}{5^{2\delta+1}} = \sum_{i=0}^{\infty} 5^i f_{i21}(x, y, z),$$

Table 1.

	1	θ	θ^2	θ^3	θ^4
ξ	—	$3\,\alpha$	$2\,\alpha$	α	0
ξ^2	$3\,\alpha$	$2\,\alpha$	α	0	$4\,\alpha$
ξ^3	$2\,\alpha$	α	0	$4\,\alpha$	$3\,\alpha$
η	—	0	δ	$2\,\delta$	$3\,\delta$
η^2	$3\,\delta$	$4\,\delta$	0	δ	$2\,\delta$
η^3	$2\,\delta$	$3\,\delta$	$4\,\delta$	0	δ
ζ	—	$3\,\varkappa$	$2\,\varkappa$	$\varkappa$	0
ζ^2	$3\,\varkappa$	$2\,\varkappa$	$\varkappa$	0	$4\,\varkappa$
ζ^3	$2\,\varkappa$	$\varkappa$	0	$4\,\varkappa$	$3\,\varkappa$
$\xi\,\eta$	0	δ	$2\,\delta$	$3\,\delta\,(2\,\alpha)$	α
$\xi^2\,\eta$	δ	$2\,\delta$	$3\,\delta\,(2\,\alpha)$	α	0
$\xi^3\,\eta$	$2\,\delta$	$3\,\delta\,(2\,\alpha)$	α	0	δ
$\xi\,\eta^2$	$4\,\delta\quad[\alpha]$	0	δ	$2\,\delta$	$3\,\delta\,(2\,\alpha)$
$\xi\,\eta^3$	$3\,\delta\,(2\,\alpha)$	$4\,\delta\quad[\alpha]$	0	δ	$2\,\delta$
$\xi^2\,\eta^2$	0	δ	$2\,\delta$	$3\,\delta\,(2\,\alpha)$	$4\,\delta\quad[\alpha]$

where f_{i11} and f_{i21} are polynomials with coefficients which are rational integers modulo 5. We have

$$F_{11} = x\cdot a^4 b\, 5^{4\alpha-\delta} + \binom{x}{2} p'_{12}\, 5^{4\alpha-\delta+1} + \binom{x}{3} p'_{13}\, 5^{4\alpha-\delta+2} + \binom{x}{4} p_{14}\, 5^{4\alpha-\delta+3} -$$

$$+ \cdots + y\cdot c^4 d + \binom{y}{2} q_{12}\, 5^{\delta+1} + \cdots + x\,y\, s'_{111}\, 5^{\alpha+\delta+1} + x\binom{y}{2} s_{112}\, 5^{\alpha+\delta+2} +$$

$$+ \binom{x}{2} y\, s_{121}\, 5^{2\alpha+2} + \cdots - z\cdot f^4 g\, 5^{4\varkappa-\delta} - \binom{z}{2} r'_{12}\, 5^{4\varkappa-\delta+1} -$$

$$- \binom{z}{3} r'_{13}\, 5^{4\varkappa-\delta+2} - \binom{z}{4} r_{14}\, 5^{4\varkappa-\delta+3} - \cdots .$$

$$F_{21} = x\cdot 2\, a^3 b^2\, 5^{3\alpha-2\delta} + \binom{x}{2} p'_{22}\, 5^{3\alpha-2\delta+1} + \binom{x}{3} p_{23}\, 5^{3\alpha-2\delta+2} + \cdots +$$

$$+ y\cdot 2\, c^3 d^2 + \binom{y}{2} q_{22}\, 5 + \cdots + x\,y\, s'_{211}\, 5^{\alpha+\delta+1} + x\binom{y}{2} s'_{212}\, 5^{\alpha+\delta+2} +$$

$$+ x\binom{y}{3} s_{213}\, 5^{\alpha+\delta+3} + \cdots + \binom{x}{2} y\, s'_{221}\, 5^{2\alpha+2\delta+2} + \binom{x}{2}\binom{y}{2} s_{222}\, 5^{2\alpha+3} -$$

$$+ \cdots - z\cdot 2\, f^3 g^2\, 5^{3\varkappa-2\delta} - \binom{z}{2} r'_{22}\, 5^{3\varkappa-2\delta+1} - \binom{z}{3} r_{23}\, 5^{3\varkappa-2\delta+2} - \cdots .$$

25.1. Now let us suppose that $2\varkappa > 3\delta$. Then $2\alpha > 3\delta$ and we may write

$$F_{31}(x, y, z) = \frac{F_3(x, y, z)}{5^{3\delta+1}} = \sum_{i=0}^{\infty} 5^i f_{i31}(x, y, z),$$

where f_{i31} are polynomials with coefficients which are rational integers modulo 5. We have

$$F_{31} = x \cdot 2\, a^2 b^3 5^{2\alpha-3\delta} + \binom{x}{2} p_{32} 5^{2\alpha-3\delta+1} + \cdots + y \cdot 2\, c^2 d^3 + \binom{y}{2} q'_{32} 5 +$$

$$+ \binom{y}{3} q_{33} 5^2 + \cdots + x\, y\, s'_{311} 5^{\alpha+\delta+1} + x \binom{y}{2} s'_{312} 5^{\alpha+\delta+2} + x \binom{y}{3} s'_{313} 5^{\alpha+\delta+2} +$$

$$+ x \binom{y}{4} s_{314} 5^{\alpha+\delta+4} + \cdots + \binom{x}{2} y\, s'_{321} 5^{3\alpha-2\delta+2} + \binom{x}{2}\binom{y}{2} s'_{322} 5^{2\alpha+2\delta+3} +$$

$$+ \binom{x}{2}\binom{y}{3} s_{323} 5^{2\alpha+4} + \cdots + \binom{x}{3} y\, s_{331} 5^{3\alpha-2\delta+3} + \cdots - z \cdot 2\, f^2 g^3 5^{2\varkappa-3\delta} -$$

$$- \binom{z}{2} r_{32} 5^{2\varkappa-3\delta+1} - \cdots.$$

Here $3\alpha - 2\delta + 2 > \alpha + \delta + 1$ since $2\alpha > 3\delta$. According to (5) we get

$$F_{11} = 5^{4\alpha-\delta} x H_1(x) + y K_1(y) + 5^{\alpha+\delta+1} x y L_1(x, y) - \tag{6}$$
$$- 5^{4\varkappa-\delta} z M_1(z) = 0 \quad (5),$$

$$F_{21} = 5^{3\alpha-2\delta} x H_2(x) + y K_2(y) + 5^{\alpha+\delta+1} x y L_2(x, y) - \tag{7}$$
$$- 5^{3\varkappa-2\delta} z M_2(z) = 0 \quad (5),$$

$$F_{31} = 5^{2\alpha-3\delta} x H_3(x) + y K_3(y) + 5^{\alpha+\delta+1} x y L_3(x, y) - \tag{8}$$
$$- 5^{2\varkappa-3\delta} z M_3(z) = 0 \quad (5).$$

Here

$$H_j = \sum_{i=0}^{\infty} 5^i h_{ij}(x),\ K_j = \sum_{i=0}^{\infty} 5^i k_{ij}(y),\ L_j = \sum_{i=0}^{\infty} 5^i l_{ij}(x, y) \text{ and } M_j = \sum_{i=0}^{\infty} 5^i m_{ij}(z),$$

where h_{ij}, k_{ij}, l_{ij} and m_{ij} are polynomials with coefficients which are rational integers modulo 5. Evidently, we have

$$K_j \not\equiv 0 \pmod 5, \quad M_j \not\equiv 0 \pmod 5 \qquad (j = 1, 2, 3). \tag{9}$$

Further we have

$$4\alpha - \delta > 3\alpha - 2\delta > 2\alpha - 3\delta, \quad 4\varkappa - \delta > 3\varkappa - 2\delta > 2\varkappa - 3\delta, \quad \alpha > \varkappa.$$

25.1.1. Let us first suppose that $2\varkappa - 3\delta \leqq \alpha + \delta + 1$. According to (9), it follows from (8) that $y \equiv 0 \pmod{5^{2\varkappa - 3\delta}}$. Thus we may put $y = 5^{2\varkappa - 3\delta} y_1$, where y_1 is an integer. Putting this in (7) we get

$$F_{22} = 5^{3\alpha+\delta-2\varkappa} x H_2 + y_1 K_{21} + 5^{\alpha+\delta+1} x y_1 L_{21} - 5^{\varkappa+\delta} z M_2 = 0 \quad (5), \tag{10}$$

where $K_{21} = K_2(5^{2\varkappa-3\delta} y_1)$, $L_{21} = L_2(x, 5^{2\varkappa-3\delta} y_1)$ and where the second index of F_{22} indicates that F_{21} has been divided by $5^{2\varkappa-3\delta}$. Evidently F_{22} is still a function of the same kind as F_{21}, i.e.

$$F_{22} = \sum_{i=0}^{\infty} 5^i f_{i22},$$

where f_{i22} are polynomials with coefficients which are rational integers modulo 5. Since

$$\alpha + \delta + 1 > \varkappa + \delta \quad \text{and} \quad 3\alpha + \delta - 2\varkappa = \alpha + \delta + 2(\alpha - \varkappa) > \varkappa + \delta$$

it follows from (10) that $y_1 \equiv 0 \pmod{5^{\varkappa+\delta}}$. Thus we may put $y_1 = 5^{\varkappa+\delta} y_2$, where y_2 is an integer. Putting $y = 5^{3\varkappa-2\delta} y_2$ in (6) we get

$$F_{12} = 5^{4\alpha+\delta-3\varkappa} x H_1 + y_2 K_{12} + 5^{\alpha+\delta+1} x y_2 L_{12} - 5^{\varkappa+\delta} z M_1 = 0 \quad (5), \tag{11}$$

where the notations are analogous to those we have used above. From (11) we get $y_2 = 5^{\varkappa+\delta} y_3$, where y_3 is an integer. Using this we get from (6)–(8)

$$F_{13} = 5^{4(\alpha-\varkappa)} x H_1 + y_3 K_{13} + 5^{\alpha+\delta+1} x y_3 L_{13} - z M_1 = 0 \quad (5), \tag{12}$$

$$F_{23} = 5^{3(\alpha-\varkappa)} x H_2 + 5^{\varkappa+\delta} y_3 K_{23} + 5^{\alpha+\delta+1+\varkappa+\delta} x y_3 L_{23} - z M_2 = 0 \quad (5), \tag{13}$$

$$F_{33} = 5^{2(\alpha-\varkappa)} x H_3 + 5^{2(\varkappa+\delta)} y_3 K_{33} + 5^{\alpha+\delta+1+2(\varkappa+\delta)} x y_3 L_{33} - z M_3 = 0 \quad (5). \tag{14}$$

The notations are analogous to those we have used above.

Let $\alpha - \varkappa \leqq \varkappa + \delta$. Putting $k = 2(\alpha - \varkappa)$ we get

$$\frac{\partial(F_{13}^{(k)}, F_{23}^{(k)}, F_{33}^{(k)})}{\partial(x, y_3, z)} \equiv -4 a^2 b^3 c^4 d f^3 g^2 5^k \not\equiv 0 \pmod{5^{k+1}}$$

since $\alpha + \delta + 1 + \varkappa + \delta > 2\varkappa + 2\delta + 1 > 2(\varkappa + \delta) \geqq 2(\alpha - \varkappa)$.

If $\alpha - \varkappa > \varkappa + \delta$ it follows from (14), according to (9), that $z = 5^{2(\varkappa+\delta)} z_1$, where z_1 is an integer. Hence we get from (13)

$$F_{24} = 5^{3(\alpha-\varkappa)-(\varkappa+\delta)} x H_2 + y_3 K_{23} + 5^{\alpha+\delta+1} x y_3 L_{23} - \\ - 5^{\varkappa+\delta} z_1 M_{21} = 0 \quad (5), \tag{15}$$

where $F_{24} = \frac{F_{23}}{5^{\varkappa+\delta}}$ and $M_{21} = M_2(5^{2(\varkappa+\delta)} z_1)$. Since $3(\alpha-\varkappa) - (\varkappa+\delta) > \varkappa + \delta$ it follows from (15) that $y_3 = 5^{\varkappa+\delta} y_4$, where y_4 is an integer. Using this we get from (12)

$$F_{14} = 5^{4(\alpha-\varkappa)-(\varkappa+\delta)} x H_1 + y_4 K_{14} + 5^{\alpha+\delta+1} x y_4 L_{14} - 5^{\varkappa+\delta} z_1 M_{11} = 0 \quad (5).$$

Hence $y_4 = 5^{\varkappa+\delta} y_5$, where y_5 is an integer. Evidently, we get from (12)–(14) the following pentadic system

$$\left| \begin{array}{l} F_{15} = 5^{4(\alpha-\varkappa)-2(\varkappa+\delta)} x H_1 + y_5 K_{15} + 5^{\alpha+\delta+1} x y_5 L_{15} - z_1 M_{11} = 0 \quad (5), \\ F_{25} = 5^{3(\alpha-\varkappa)-2(\varkappa+\delta)} x H_2 + 5^{\varkappa+\delta} y_5 K_{25} + 5^{\alpha+\delta+1+\varkappa+\delta} x y_5 L_{25} - \\ \qquad - z_1 M_{21} = 0 \quad (5), \\ F_{35} = 5^{2(\alpha-\varkappa)-2(\varkappa+\delta)} x H_3 + 5^{2(\varkappa+\delta)} y_5 K_{35} + 5^{\alpha+\delta+1+2(\varkappa+\delta)} x y_5 L_{35} - \\ \qquad - z_1 M_{31} = 0 \quad (5), \end{array} \right.$$

Here the notations are again analogous to those we have used before. If $\alpha - \varkappa \leqq 2(\varkappa+\delta)$ then

$$2(\alpha-\varkappa) - 2(\varkappa+\delta) \leqq 2(\varkappa+\delta) < \alpha+\delta+1+2(\varkappa+\delta)$$

and

$$\alpha + \delta + 1 + \varkappa + \delta - 2\varkappa - 2\delta + 1 > 2(\varkappa+\delta).$$

For $k = 2(\alpha-\varkappa) - 2(\varkappa+\delta)$ we thus get

$$\frac{\partial(F_{15}^{(k)}, F_{25}^{(k)}, F_{35}^{(k)})}{\partial(x, y_5, z_1)} \equiv -4a^2 b^3 c^4 d f^3 g^2 5^k \not\equiv 0 \pmod{5^{k+1}}. \tag{16}$$

If $\alpha - \varkappa > 2(\varkappa+\delta)$, the same argument may be repeated. In this way, after a finite number of operations, we can always find a k such that a congruence relation analogous to (16) holds. Thus we have proved that the pentadic system (6)–(8) has only a finite number of integral solutions if $2\varkappa - 3\delta \leqq \alpha + \delta + 1$.

25.1.2. Let us now suppose that $2\varkappa - 3\delta > \alpha + \delta + 1$. We consider the system (6)–(8). It follows from (8) that $y = 5^{\alpha+\delta+1} y_1$, where y_1 is an integer. Using this we get from (8)

$$F_{32} = 5^{\alpha-4\delta-1} x H_3 + y_1 K_{31} + 5^{\alpha+\delta+1} x y_1 L_{31} - \\ - 5^{2\varkappa-4\delta-\alpha-1} z M_3 = 0 \quad (5). \tag{17}$$

Since $\alpha + \delta - 1 > \alpha - 4\delta - 1 > 2\varkappa - 4\delta - \alpha - 1$ it follows from (17) that $y_1 = 5^{2\varkappa-4\delta-\alpha-1} y_2$, where y_2 is an integer, and we may put $y = 5^{2\varkappa-3\delta} y_2$. As before we get from (7)

$$F_{22} = 5^{3\alpha+\delta-2\varkappa} x H_2 + y_2 K_{22} + 5^{\alpha+\delta+1} x y_2 L_{22} - 5^{\varkappa+\delta} z M_2 = 0 \quad (5).$$

Hence $y_2 = 5^{\varkappa-\delta} y_3$, where y_3 is an integer. Using this we get from (6)

$$F_{12} = 5^{4\alpha-\delta-3\varkappa} x H_1 + y_3 K_{13} + 5^{\alpha+\delta+1} x y_3 L_{13} - 5^{\varkappa+\delta} z M_1 = 0 \quad (5).$$

Hence $y_3 = 5^{\varkappa-\delta} y_4$, where y_4 is an integer. Hence we get from (6)–(8) the following pentadic system:

$$\left| \begin{array}{l} F_{13} = 5^{4(\alpha-\varkappa)} x H_1 + y_4 K_{14} + 5^{\alpha+\delta+1} x y_4 L_{14} - z M_1 = 0 \quad (5). \\ F_{23} = 5^{3(\alpha-\varkappa)} x H_2 + 5^{\varkappa+\delta} y_4 K_{24} + 5^{\alpha+\delta+1+\varkappa+\delta} x y_4 L_{24} - \\ \qquad - z M_2 = 0 \quad (5). \\ F_{33} = 5^{2(\alpha-\varkappa)} x H_3 + 5^{2(\varkappa+\delta)} y_4 K_{34} + 5^{\alpha+\delta+1+2(\varkappa+\delta)} x y_4 L_{34} - \\ \qquad - z M_3 = 0 \quad (5). \end{array} \right. \tag{18}$$

System (18) is identical in form with system (12)–(14), and exactly the same argument may now be employed as for that system. Thus we have treated the case $2\varkappa > 3\delta$ completely.

25.2. Now let us suppose that $2\varkappa \leqq 3\delta$. We distinguish two cases $2\alpha > 3\delta \geqq 2\varkappa$ and $3\delta \geqq 2\alpha > 2\varkappa$.

25.2.1. Let us first consider the case $2\alpha > 3\delta \geqq 2\varkappa$. From (4) we get

$$F_{31}(x, y, z) = \frac{F_3(x, y, z)}{5^{2\varkappa-1}} = \sum_{i=0}^{\infty} 5^i f_{i31}(x, y, z),$$

where f_{i31} are polynomials with coefficients which are rational integers modulo 5. We may write

(19) $$F_{31} = 5^{2(\alpha-\varkappa)} x H_3 + 5^{3\delta-2\varkappa} y K_3 + 5^{\alpha+4\delta-2\varkappa+1} x y L_3 - z M_3 = 0 \quad (5),$$

where H_3, K_3, L_3 and M_3 are functions of the same type as the corresponding functions in (8).

We have to regard the pentadic system formed by (6), (7) and (19):

(20) $$\begin{cases} F_{11} = 5^{4\alpha-\delta} x H_1 + y K_1 + 5^{\alpha+\delta+1} x y L_1 - 5^{4\varkappa-\delta} z M_1 = 0 \quad (5), \\ F_{21} = 5^{3\alpha-2\delta} x H_2 + y K_2 + 5^{\alpha+\delta+1} x y L_2 - 5^{3\varkappa-2\delta} z M_2 = 0 \quad (5), \\ F_{31} = 5^{2(\alpha-\varkappa)} x H_3 + 5^{3\delta-2\varkappa} y K_3 + 5^{\alpha+4\delta-2\varkappa+1} x y L_3 - z M_3 = 0 \quad (5). \end{cases}$$

Since $3\delta \geqq 2\varkappa$ then

$$\alpha + \delta + 1 = \alpha + 3\delta + 1 - 2\delta \geqq \alpha + 2\varkappa + 1 - 2\delta > 3\varkappa - 2\delta.$$

From the equation $F_{21} = 0$ (5) we get $y = 5^{3\varkappa-2\delta} y_1$, where y_1 is an integer. Substituting this in $F_{11} = 0$ (5) we get

$$F_{12} = 5^{4\alpha+\delta-3\varkappa} x H_1 + y_1 K_{11} + 5^{\alpha+\delta+1} x y_1 L_{11} - 5^{\varkappa+\delta} z M_1 = 0 \quad (5),$$

where the notations are analogous to those we have used before. It follows from the equation $F_{12} = 0$ (5) that $y_1 = 5^{\varkappa+\delta} y_2$, where y_2 is an integer. Using this we get from system (20)

$$\begin{cases} F_{13} = 5^{4(\alpha-\varkappa)} x H_1 + y_2 K_{12} + 5^{\alpha+\delta+1} x y_2 L_{12} - z M_1 = 0 \quad (5), \\ F_{23} = 5^{3(\alpha-\varkappa)} x H_2 + 5^{\varkappa+\delta} y_2 K_{22} + 5^{\alpha+\delta+1+\varkappa+\delta} x y_2 L_{22} - z M_2 = 0 \quad (5), \\ F_{33} = 5^{2(\alpha-\varkappa)} x H_3 + 5^{2(\varkappa+\delta)} y_2 K_{32} + 5^{\alpha+\delta+1+2(\varkappa+\delta)} x y_2 L_{32} - z M_3 = 0 \quad (5). \end{cases}$$

However, this system is identical in form with system (12)–(14) in no. **25.1.1.** and exactly the same reasoning as in that no. may now be applied. We have thus treated the case $2\alpha > 3\delta \geqq 2\varkappa$.

25.2.2. Next let us consider the case $3\delta \geqq 2\alpha > 2\varkappa$. From (4) we get

(21) $$F_{31}(x, y, z) = \frac{F_3(x, y, z)}{5^{2\varkappa+1}} = \sum_{i=0}^{\infty} 5^i f_{i31}(x, y, z),$$

where f_{i31} are polynomials with coefficients which are rational integers modulo 5. We have to investigate the pentadic system formed by (6), (7) and (21)

$$(22)\quad \begin{cases} F_{11} = 5^{4\alpha-\delta}\, x H_1 + y K_1 + 5^{\alpha+\delta+1}\, x y L_1 - 5^{4\varkappa-\delta}\, z M_1 = 0 \quad (5), \\ F_{21} = 5^{3\alpha-2\delta}\, x H_2 + y K_2 + 5^{\alpha+\delta+1} x y L_2 - 5^{3\varkappa-2\delta}\, z M_2 = 0 \quad (5), \\ F_{31} = 5^{2(\alpha-\varkappa)}\, x H_3 + 5^{3\delta-2\varkappa}\, y K_3 + 5^{3\alpha+\delta-2\varkappa+1} x y L_3 - z M_3 = 0 \quad (5). \end{cases}$$

Here H_3, K_3, L_3 and M_3 are functions of the same type as the corresponding functions in (8). We have $4\alpha-\delta > \alpha+\delta+1 > 3\alpha-2\delta$ and $3\alpha+\delta-2\varkappa+1 > 2(\alpha-\varkappa)$. Hence we get for $k = 2\alpha+\varkappa-2\delta$.

$$\frac{\partial (F_{11}^{(k)}, F_{21}^{(k)}, F_{31}^{(k)})}{\partial (x, y, z)} \equiv -4\, a^2 b^3 c^4 d f^3 g^2 5^k \equiv 0 \pmod{5^{k+1}}.$$

According to Theorem 3, the pentadic system (22) has only a finite number of integral solutions.

26. Now let us investigate the case 1 b) in no. **24**, where $\alpha = \varkappa > \delta > 0$. Substituting $\varkappa = \alpha$ in (1), (2), (3) and (4) we find that we may write

$$F_{11}(x, y, z) = \frac{F_1(x, y, z)}{5^{\delta+1}} = \sum_{i=0}^{\infty} 5^i f_{i11}(x, y, z),$$

$$F_{21}(x, y, z) = \frac{F_2(x, y, z)}{5^{2\delta+1}} = \sum_{i=0}^{\infty} 5^i f_{i21}(x, y, z),$$

where f_{i11} and f_{i21} are polynomials with coefficients which are rational integers modulo 5.

26.1. Let us first suppose that $2\alpha > 3\delta$. Then

$$F_{31}(x, y, z) = \frac{F_3(x, y, z)}{5^{3\delta+1}} = \sum_{i=0}^{\infty} 5^i f_{i31}(x, y, z)$$

where f_{i31} are polynomials of the same type as f_{i11} and f_{i21}. We have

$$(23)\quad F_{11} = 5^{4\alpha-\delta}\, U_1(x, z) + y K_1(y) + 5^{\alpha+\delta+1}\, x y L_1(x, y) = 0 \quad (5),$$

$$(24)\quad F_{21} = 5^{3\alpha-2\delta}\, U_2(x, z) + y K_2(y) + 5^{\alpha+\delta+1}\, x y L_2(x, y) = 0 \quad (5),$$

$$(25)\quad F_{31} = 5^{2\alpha-3\delta}\, U_3(x, z) + y K_3(y) + 5^{\alpha+\delta+1}\, x y L_3(x, y) = 0 \quad (5).$$

Here $U_j = \sum_{i=0}^{\infty} 5^i u_{ij}(x, z)$, $K_j = \sum_{i=0}^{\infty} 5^i k_{ij}(y)$ and $L_j = \sum_{i=0}^{\infty} 5^i l_{ij}(x, y)$ where u_{ij}, k_{ij} and l_{ij} are polynomials with coefficients which are rational integers modulo 5. We have

$$u_{01} = a^4 b\, x - f^4 g\, z, \qquad k_{01} = c^4 d,$$

$$u_{02} = 2\, a^3 b^2 x - 2 f^3 g^2 z, \qquad k_{02} = 2 c^3 d^2,$$

$$u_{03} = 2\, a^2 b^3 x - 2 f^2 g^3 z, \qquad k_{03} = 2 c^2 d^3.$$

Hence

$$K_j \equiv 0 \pmod 5 \qquad (j = 1, 2, 3). \tag{26}$$

26.1.1. Let us consider the case where $2\alpha - 3\delta \leqq \alpha + \delta + 1$. It follows from (25), according to (26), that $y \equiv 0 \pmod{5^{2\alpha - 3\delta}}$. Putting $y = 5^{2\alpha - 3\delta} y_1$ and using notations analogous to those we have used before we get from (24)

$$F_{22} = 5^{\alpha+\delta} U_2 + y_1 K_{21} + 5^{\alpha+\delta+1} x\, y_1 L_{21} = 0 \quad (5),$$

which gives $y_1 = 5^{\alpha+\delta} y_2$, where y_2 is an integer. Hence we get from (23)

$$F_{12} = 5^{\alpha+\delta} U_1 + y_2 K_{12} + 5^{\alpha+\delta+1} x\, y_2 L_{12} = 0 \quad (5).$$

Thus $y_2 = 5^{\alpha+\delta} y_3$, where y_3 is an integer. From (23)–(25) we get the system

$$\begin{cases} F_{13} = U_1 + y_3 K_{13} + 5^{\alpha+\delta+1} x\, y_3 L_{13} = 0 \quad (5), \\ F_{23} = U_2 + 5^{\alpha+\delta} y_3 K_{23} + 5^{2(\alpha+\delta)+1} x\, y_3 L_{23} = 0 \quad (5), \\ F_{33} = U_3 + 5^{2(\alpha+\delta)} y_3 K_{33} + 5^{3(\alpha+\delta)+1} x\, y_3 L_{33} = 0 \quad (5). \end{cases} \tag{27}$$

If $\Delta_1 = a g - b f \equiv 0 \pmod 5$ we get from (27)

$$\frac{\partial (F_{13}^{(0)}, F_{23}^{(0)}, F_{33}^{(0)})}{\partial (x, y_3, z)} = 4\, a^2 b^2 c^4 d\, f^2 g^2\, \Delta_1 \equiv 0 \pmod 5.$$

Hence, according to Theorem 3, the pentadic system (27) has only a finite number of integral solutions under the above assumptions.

Now let Δ_1 be divisible by 5^μ (μ natural number) but not by $5^{\mu+1}$. According to Lemma 2 we may write

$$a = f + a_1 5^\mu, \quad b = g + b_1 5^\mu, \tag{28}$$

where a_1 and b_1 are integers. $\Delta_1 = \Delta_1' 5^\mu$, where $\Delta_1' = a_1 g - b_1 f \not\equiv 0 \pmod 5$. Substituting (28) in (27) we get

$$\left\{\begin{aligned} &F_{13} = (x-z) N_1 + 5^\mu T_1 + y_3 K_{13} + 5^{\alpha+\delta+1} x y_3 L_{13} = 0 \quad (5), \\ &F_{23} = (x-z) N_2 + 5^\mu T_2 + 5^{\alpha+\delta} y_3 K_{23} + 5^{2(\alpha+\delta)+1} x y_3 L_{23} = 0 \quad (5), \\ &F_{33} = (x-z) N_3 + 5^\mu T_3 + 5^{2(\alpha+\delta)} y_3 K_{33} + 5^{3(\alpha+\delta)+1} x y_3 L_{33} = 0 \quad (5). \end{aligned}\right. \tag{29}$$

Here $N_j = \sum_{i=0}^{\infty} 5^i n_{ij}(x, z)$ and $T_j = \sum_{i=0}^{\infty} 5^i t_{ij}(x)$, where n_{ij} and t_{ij} are polynomials with coefficients which are rational integers modulo 5. We have

$$n_{01} = f^4 g, \qquad t_{01} = \quad 4 f^3 \Delta_1' x,$$

$$n_{02} = 2 f^3 g^2, \quad t_{02} = -4 f^2 g \Delta_1' x,$$

$$n_{03} = 2 f^2 g^3, \quad t_{03} = \quad 4 f g^2 \Delta_1' x.$$

Hence $N_j \not\equiv 0 \pmod 5$. If $\mu < \alpha + \delta$, it follows from the relation $F_{23} = 0$ (5) that $x - z \equiv 0 \pmod{5^\mu}$. Hence for $k = \mu$ we get from (29)

$$\frac{\partial (F_{13}^{(k)}, F_{23}^{(k)}, F_{33}^{(k)})}{\partial (x, y_3, z)} \equiv -16 c^4 d f^4 g^4 \Delta_1' 5^k \not\equiv 0 \pmod{5^{k+1}}. \tag{30}$$

If $\alpha + \delta \leqq \mu < 2(\alpha + \delta)$, it follows from the relation $F_{23} = 0$ (5) that $x - z \equiv 0 \pmod{5^{\alpha+\delta}}$. According to the relation $F_{13} = 0$ (5) we get $y_3 = 5^{\alpha+\delta} y_4$, where y_4 is an integer. Thus (29) may be written

$$\left\{\begin{aligned} &F_{13} = (x-z) N_1 + 5^\mu T_1 + 5^{\alpha+\delta} y_4 K_{14} + 5^{2(\alpha+\delta)+1} x y_4 L_{14} = 0 \quad (5), \\ &F_{23} = (x-z) N_2 + 5^\mu T_2 + 5^{2(\alpha+\delta)} y_4 K_{24} + 5^{3(\alpha+\delta)+1} x y_4 L_{24} = 0 \quad (5), \\ &F_{33} = (x-z) N_3 + 5^\mu T_3 + 5^{3(\alpha+\delta)} y_4 K_{34} + 5^{4(\alpha+\delta)+1} x y_4 L_{34} = 0 \quad (5). \end{aligned}\right.$$

Evidently $x - z \equiv 0 \pmod{5^\mu}$. Hence, for $k = \alpha + \delta + \mu$ we get again a relation of the form (30), in fact we have only to substitute y_4

for y_3. If $\mu \geqq 2(\alpha+\delta)$, the same argument may be repeated. In this way after a finite number of operations we can always find a k such that a congruence relation analogous to (30) holds. We have thus proved that the pentadic system (23)–(25) has only a finite number of integral solutions under the above assumptions.

26.1.2. Let us next consider system (23)–(25) when

$$2\alpha < 3\delta + \alpha + \delta + 1.$$

It follows from (25) that $y = 5^{\alpha+\delta+1} y_1$, where y_1 is an integer. Hence we get from (25)

$$F_{32} = 5^{\alpha-4\delta-1} U_3 + y_1 K_{31} + 5^{\alpha+\delta+1} x y_1 L_{31} = 0 \quad (5).$$

$\alpha - 4\delta - 1 > 0$, and we get $y_1 = 5^{\alpha-4\delta-1} y_2$, where y_2 is an integer. By (24) we get

$$F_{22} = 5^{\alpha+\delta} U_2 + y_2 K_{22} + 5^{\alpha+\delta+1} x y_2 L_{22} = 0 \quad (5).$$

Thus $y_2 = 5^{\alpha+\delta} y_3$, where y_3 is an integer. Substituting this in (23) we find

$$F_{12} = 5^{\alpha+\delta} U_1 + y_3 K_{13} + 5^{\alpha+\delta+1} x y_3 L_{13} = 0 \quad (5),$$

and $y_3 = 5^{\alpha+\delta} y_4$, where y_4 is an integer. From (23)–(25) we get the system

$$\begin{cases} F_{13} = U_1 + y_4 K_{14} + 5^{\alpha+\delta+1} x y_4 L_{14} = 0 \quad (5), \\ F_{23} = U_2 + 5^{\alpha+\delta} y_4 K_{24} + 5^{2(\alpha+\delta)+1} x y_4 L_{24} = 0 \quad (5), \\ F_{33} = U_3 + 5^{2(\alpha+\delta)} y_4 K_{34} + 5^{3(\alpha+\delta)+1} x y_4 L_{34} = 0 \quad (5). \end{cases}$$

However, this system is identical in form with system (27) and exactly the same argument may now be employed as for that system. We have thus treated the case $2\alpha > 3\delta$ completely.

26.2. Now let us suppose that $2\alpha \leqq 3\delta$. We find that

$$(31) \qquad F_{31}(x, y, z) = \frac{F_3(x, y, z)}{5^{2\alpha+1}} - \sum_{i=0}^{\infty} 5^i f_{i31}(x, y, z) = 0 \quad (5),$$

where f_{i31} are polynomials with coefficients which are rational integers modulo 5. We have to consider the system formed by (23), (24) and (31):

$$(32)\qquad \begin{cases} F_{11} = 5^{4\alpha-\delta}\, U_1 + y\, K_1 + 5^{\alpha+\delta+1}\, x\, y\, L_1 = 0 \quad (5), \\ F_{21} = 5^{3\alpha-2\delta}\, U_2 + y\, K_2 + 5^{\alpha+\delta+1}\, x\, y\, L_2 = 0 \quad (5), \\ F_{31} = U_3 + 5^{3\delta-2\alpha}\, y\, K_3 + 5^{\alpha+\delta+1}\, x\, y\, L_3 = 0 \quad (5). \end{cases}$$

U_3, K_3 and L_3 are functions of the same kind as the corresponding functions in (25). We have as before $u_{03} = 2\,a^2 b^3 x - 2\,f^2 g^3 z$ and $k_{03} = 2\,c^2 d^3$. Since $\alpha + \delta + 1 > 3\,\delta + \alpha - 2\,\delta \geqq 3\,\alpha - 2\,\delta$, it follows from the relation $F_{21} = 0$ (5) that $y = 5^{3\alpha-2\delta}\, y_1$, where y_1 is an integer. Thus we get from the relation $F_{11} = 0$ (5)

$$F_{12} = 5^{\alpha+\delta}\, U_1 + y_1\, K_{11} + 5^{\alpha+\delta+1}\, x\, y_1\, L_{11} = 0 \quad (5),$$

which gives $y_1 = 5^{\alpha+\delta}\, y_2$, where y_2 is an integer. Hence, system (32) may be written

$$\begin{cases} F_{13} = U_1 + y_2\, K_{12} + 5^{\alpha-\delta+1}\, x\, y_2\, L_{12} = 0 \quad (5), \\ F_{23} = U_2 + 5^{\alpha+\delta}\, y_2\, K_{22} + 5^{2(\alpha+\delta)+1}\, x\, y_2\, L_{22} = 0 \quad (5), \\ F_{33} = U_3 + 5^{2(\alpha+\delta)}\, y_2\, K_{32} + 5^{5\alpha+1}\, x\, y_2\, L_{32} = 0 \quad (5). \end{cases}$$

This system is analogous to system (27) and it is evident that exactly the same reasoning as is no. **26.1.1.** may be applied. We have thus treated the case $2\,\alpha \leqq 3\,\delta$ completely.

27. Let us consider the case 1 c) in no. **24**, where $\alpha > \varkappa = \delta > 0$.

Let us first suppose that $2\,\alpha > 3\,\delta$. If we put $\varkappa = \delta$ in system (20), then we get the pentadic system in this case. It is easily seen that the same argument may be employed as for system (20).

Let us next suppose that $2\,\alpha \leqq 3\,\delta$. If we put $\varkappa = \delta$ in system (22), then we get the pentadic system in this case. For $k = 2\,\alpha - \delta$ we get the same congruence relation as in no. **25.2.2.**

We have thus treated the case $\alpha > \varkappa = \delta > 0$ completely.

28. Let $\alpha = \varkappa = \delta > 0$. Putting $\varkappa = \delta$ and $\alpha = \delta$ in (1), (2), (3) and (4) we get

$$\begin{cases} F_{11}(x, y, z) = \dfrac{F_1(x, y, z)}{5^{\delta+1}} = \sum_{i=0}^{\infty} 5^i f_{i11}(x, y, z) = 0 \quad (5), \\ F_{21}(x, y, z) = \dfrac{F_2(x, y, z)}{5^{2\delta+1}} = \sum_{i=0}^{\infty} 5^i f_{i21}(x, y, z) = 0 \quad (5), \\ F_{31}(x, y, z) = \dfrac{F_3(x, y, z)}{5^{2\delta+1}} = \sum_{i=0}^{\infty} 5^i f_{i31}(x, y, z) = 0 \quad (5), \end{cases}$$

where f_{i11}, f_{i21} and f_{i31} are polynomials with coefficients which are rational integers modulo 5. Using the same notations as before, we may write

$$(33) \qquad \begin{cases} F_{11} = 5^{3\delta} U_1 + y K_1 + 5^{2\delta+1} x y L_1 = 0 \quad (5), \\ F_{21} = 5^{\delta} U_2 + y K_2 + 5^{2\delta+1} x y L_2 = 0 \quad (5), \\ F_{31} = U_3 + 5^{\delta} y K_3 + 5^{2\delta+1} x y L_3 = 0 \quad (5). \end{cases}$$

However, if we put $\alpha = \delta$ in system (32), then system (33) is identical with system (32). The same reasoning as for system (32) may be applied.

We have thus treated the case 1) in no. **24** where $\alpha \geqq \varkappa \geqq \delta > 0$ completely.

29. Let us investigate the case 2 a) in no. **24**, where $\alpha > \delta > \varkappa \geqq 0$.

29.1. Let us first consider the case $2\alpha > 3\delta$. By (4) and (5) in no. **24** we get

$$(34) \qquad \begin{cases} F_1 = 5^{4\alpha+1} x H_1(x) + 5^{\delta+1} y K_1(y) + 5^{\alpha+2\delta+2} x y L_1(x, y) - \\ \qquad - 5^{1\varkappa+1} z M_1(z) = 0 \quad (5), \\ F_2 = 5^{3\alpha+1} x H_2(x) + 5^{2\delta+1} y K_2(y) + 5^{\alpha+3\delta+2} x y L_2(x, y) - \\ \qquad - 5^{3\varkappa+1} z M_2(z) = 0 \quad (5), \\ F_3 = 5^{2\alpha+1} x H_3(x) + 5^{3\delta+1} y K_3(y) + 5^{\alpha+4\delta+2} x y L_3(x, y) - \\ \qquad - 5^{2\varkappa+1} z M_3(z) = 0 \quad (5). \end{cases}$$

Here H_j, K_j, L_j and M_j $(j=1, 2, 3)$ are the same functions as in system (6)–(8). We have

$$h_{01}=a^4 b, \qquad k_{01}=c^4 d, \qquad m_{01}=f^4 g,$$

$$h_{02}=2 a^3 b^2, \quad k_{02}=2 c^3 d^2, \quad m_{02}=2 f^3 g^2,$$

$$h_{03}=2 a^2 b^3, \quad k_{03}=2 c^2 d^3, \quad m_{03}=2 f^2 g^3.$$

and (9) holds.

29.1.1. If $\delta \geqq 4\varkappa$ it follows from (34) that

$$(35)\quad \begin{cases} F_{11}=5^{4(\alpha-\varkappa)} x H_1+5^{\delta-4\varkappa} y K_1+5^{\alpha+2\delta-4\varkappa+1} x y L_1-z M_1=0 \quad (5), \\ F_{21}=5^{3(\alpha-\varkappa)} x H_2+5^{2\delta-3\varkappa} y K_2+5^{\alpha+3\delta-3\varkappa+1} x y L_2-z M_2=0 \quad (5), \\ F_{31}=5^{2(\alpha-\varkappa)} x H_3+5^{3\delta-2\varkappa} y K_3+5^{\alpha+4\delta-2\varkappa+1} x y L_3-z M_3=0 \quad (5). \end{cases}$$

Since $2(\alpha-\varkappa)>3\delta-2\varkappa$ and $\alpha+4\delta-2\varkappa+1>3\delta-2\varkappa$ it follows from the relation $F_{31}=0$ (5) that $z=5^{3\delta-2\varkappa} z_1$, where z_1 is an integer. Substituting this in the relation $F_{21}=0$ (5) we get

$$F_{22}=5^{3\alpha-2\delta} x H_2+y K_2+5^{\alpha+\delta+1} x y L_2-5^{\varkappa+\delta} z_1 M_{21}=0 \quad (5).$$

Since $3\alpha-2\delta \geqq \alpha+\delta+1>\varkappa+\delta$ it follows from $F_{22}=0$ (5) that $y=5^{\varkappa+\delta} y_1$, where y_1 is an integer. Hence we get from $F_{11}=0$ (5)

$$F_{12}=5^{4\alpha-3\delta} x H_1+y K_1+5^{\alpha+\delta+1} x y L_1-5^{2(\varkappa+\delta)} z_1 M_{11}=0 \quad (5),$$

$$F_{13}=5^{4\alpha-2\delta-\varkappa} x H_1+y_1 K_{11}+5^{\alpha+\delta+1} x y_1 L_{11}-5^{\varkappa+\delta} z_1 M_{11}=0 \quad (5).$$

Since $4\alpha-2\delta-\varkappa>\varkappa+\delta$ it follows from $F_{13}=0$ (5) that $y_1=5^{\varkappa+\delta} y_2$, where y_2 is an integer. Thus we get from (35) the following pentadic system

$$(36)\quad \begin{cases} F_{14}=5^{2(\alpha-\varkappa)+2\alpha-3\delta} x H_1+y_2 K_{12}+5^{\alpha+\delta+1} x y_2 L_{12}- \\ \qquad -z_1 M_{11}=0 \quad (5), \\ F_{24}=5^{\alpha-\varkappa+2\alpha-3\delta} x H_2+5^{\varkappa+\delta} y_2 K_{22}+5^{\alpha+\delta+1+\varkappa+\delta} x y_2 L_{22}- \\ \qquad -z_1 M_{21}=0 \quad (5), \\ F_{34}=5^{2\alpha-3\delta} x H_3+5^{2(\varkappa+\delta)} y_2 K_{32}+5^{\alpha+\delta+1+2(\varkappa+\delta)} x y_2 L_{32}- \\ \qquad -z_1 M_{31}=0 \quad (5). \end{cases}$$

If $2\alpha - 3\delta \leqq 2(\varkappa + \delta)$ we put $k = 2\alpha - 3\delta$ and get from (36)

$$\frac{\partial(F_{14}^{(k)}, F_{24}^{(k)}, F_{34}^{(k)})}{\partial(x, y_2, z_1)} \equiv -4\, a^2 b^3 c^4 d f^3 g^2 5^k \not\equiv 0 \pmod{5^{k+1}}.$$

If $2\alpha - 3\delta > 2(\varkappa + \delta)$ it follows from $F_{34} = 0$ (5) that $z_1 = 5^{2(\varkappa+\delta)} z_2$, where z_2 is an integer. Substituting this in $F_{24} = 0$ (5) we get

$$F_{25} \quad 5^{\alpha - \varkappa + 2\alpha - 3\delta - (\varkappa+\delta)} x H_2 + y_2 K_{22} + 5^{\alpha+\delta+1} x y_2 L_{22} \quad 5^{\varkappa+\delta} z_2 M_{22} \quad 0 \quad (5),$$

which gives $y_2 - 5^{\varkappa+\delta} y_3$, where y_3 is an integer. From $F_{14} = 0$ (5) we get

$$F_{15} = 5^{2(\alpha-\varkappa)+2\alpha-3\delta-(\varkappa+\delta)} x H_1 + y_3 K_{13} + 5^{\alpha+\delta+1} x y_3 L_{13} -$$
$$5^{\varkappa+\delta} z_2 M_{12} \quad 0 \quad (5),$$

which gives $y_3 = 5^{\varkappa+\delta} y_4$, where y_4 is an integer. Hence system (36) may be written

$$(37) \quad \begin{cases} F_{16} = 5^{2(\alpha-\varkappa)+2\alpha-3\delta-2(\varkappa+\delta)} x H_1 + y_4 K_{14} + 5^{\alpha+\delta+1} x y_4 L_{14} - \\ \qquad\qquad z_2 M_{12} \quad 0 \quad (5), \\ F_{26} = 5^{\alpha-\varkappa+2\alpha-3\delta-2(\varkappa+\delta)} x H_2 + 5^{\varkappa+\delta} y_4 K_{24} + 5^{\alpha+\delta+1+\varkappa+\delta} x y_4 L_{24} - \\ \qquad\qquad z_2 M_{22} \quad 0 \quad (5), \\ F_{36} = 5^{2\alpha-3\delta-2(\varkappa+\delta)} x H_3 + 5^{2(\varkappa+\delta)} y_4 K_{34} + 5^{\alpha+\delta+1+2(\varkappa+\delta)} x y_4 L_{34} - \\ \qquad\qquad - z_2 M_{32} = 0 \quad (5). \end{cases}$$

If $2(\varkappa + \delta) < 2\alpha - 3\delta \leqq 4(\varkappa + \delta)$ we put $k = 2\alpha - 3\delta - 2(\varkappa + \delta)$ and get from (37)

$$(38) \quad \frac{\partial(F_{16}^{(k)}, F_{26}^{(k)}, F_{36}^{(k)})}{\partial(x, y_4, z_2)} \equiv -4\, a^2 b^3 c^4 d f^3 g^2 5^k \not\equiv 0 \pmod{5^{k+1}}.$$

If $2\alpha - 3\delta > 4(\varkappa + \delta)$ the same argument may be repeated. In this way, after a finite number of operations, we can always find a k such that a congruence relation analogous to (38) holds. According to Theorem 3, the pentadic system (34) has thus only a finite number of integral solutions under the above assumptions.

Here H_j, K_j, L_j and M_j $(j=1, 2, 3)$ are the same functions as in system (6)–(8). We have

$$h_{01}=a^4 b, \qquad k_{01}=c^4 d, \qquad m_{01}=f^4 g,$$

$$h_{02}=2\,a^3 b^2, \quad k_{02}=2\,c^3 d^2, \quad m_{02}=2\,f^3 g^2,$$

$$h_{03}=2\,a^2 b^3, \quad k_{03}=2\,c^2 d^3, \quad m_{03}=2\,f^2 g^3.$$

and (9) holds.

29.1.1. If $\delta \geqq 4\varkappa$ it follows from (34) that

$$(35)\quad \begin{cases} F_{11}=5^{4(\alpha-\varkappa)}\,x\,H_1+5^{\delta-4\varkappa}\,y\,K_1+5^{\alpha+2\delta-4\varkappa+1}\,x\,y\,L_1-z\,M_1=0 \quad (5), \\ F_{21}=5^{3(\alpha-\varkappa)}\,x\,H_2+5^{2\delta-3\varkappa}\,y\,K_2+5^{\alpha+3\delta-3\varkappa+1}\,x\,y\,L_2-z\,M_2=0 \quad (5), \\ F_{31}=5^{2(\alpha-\varkappa)}\,x\,H_3+5^{3\delta-2\varkappa}\,y\,K_3+5^{\alpha+4\delta-2\varkappa+1}\,x\,y\,L_3-z\,M_3=0 \quad (5). \end{cases}$$

Since $2(\alpha-\varkappa)>3\delta-2\varkappa$ and $\alpha+4\delta-2\varkappa+1>3\delta-2\varkappa$ it follows from the relation $F_{31}=0$ (5) that $z=5^{3\delta-2\varkappa}\,z_1$, where z_1 is an integer. Substituting this in the relation $F_{21}=0$ (5) we get

$$F_{22}=5^{3\alpha-2\delta}\,x\,H_2+y\,K_2+5^{\alpha+\delta+1}\,x\,y\,L_2-5^{\varkappa+\delta}\,z_1\,M_{21}=0 \quad (5).$$

Since $3\alpha-2\delta\geqq\alpha+\delta+1>\varkappa+\delta$ it follows from $F_{22}=0$ (5) that $y=5^{\varkappa+\delta}\,y_1$, where y_1 is an integer. Hence we get from $F_{11}=0$ (5)

$$F_{12}=5^{4\alpha-\delta}\,x\,H_1+y\,K_1+5^{\alpha+\delta+1}\,x\,y\,L_1-5^{2(\varkappa+\delta)}\,z_1\,M_{11}=0 \quad (5),$$

$$F_{13}=5^{4\alpha-2\delta-\varkappa}\,x\,H_1+y_1\,K_{11}+5^{\alpha+\delta+1}\,x\,y_1\,L_{11}-5^{\varkappa+\delta}\,z_1\,M_{11}=0 \quad (5).$$

Since $4\alpha-2\delta-\varkappa>\varkappa+\delta$ it follows from $F_{13}=0$ (5) that $y_1=5^{\varkappa+\delta}\,y_2$, where y_2 is an integer. Thus we get from (35) the following pentadic system

$$(36)\quad \begin{cases} F_{14}=5^{2(\alpha-\varkappa)+2\alpha-3\delta}\,x\,H_1+y_2\,K_{12}+5^{\alpha+\delta+1}\,x\,y_2\,L_{12}- \\ \qquad\qquad -z_1\,M_{11}=0 \quad (5), \\ F_{24}=5^{\alpha-\varkappa+2\alpha-3\delta}\,x\,H_2+5^{\varkappa+\delta}\,y_2\,K_{22}+5^{\alpha+\delta+1+\varkappa+\delta}\,x\,y_2\,L_{22}- \\ \qquad\qquad -z_1\,M_{21}=0 \quad (5), \\ F_{34}=5^{2\alpha-3\delta}\,x\,H_3+5^{2(\varkappa+\delta)}\,y_2\,K_{32}+5^{\alpha+\delta+1+2(\varkappa+\delta)}\,x\,y_2\,L_{32}- \\ \qquad\qquad -z_1\,M_{31}=0 \quad (5). \end{cases}$$

If $2\alpha - 3\delta \leqq 2(\varkappa + \delta)$ we put $k = 2\alpha - 3\delta$ and get from (36)

$$\frac{\partial (F_{14}^{(k)}, F_{24}^{(k)}, F_{34}^{(k)})}{\partial (x, y_2, z_1)} - 4 a^2 b^3 c^4 d f^3 g^2 5^k \equiv 0 \pmod{5^{k+1}}.$$

If $2\alpha - 3\delta > 2(\varkappa + \delta)$ it follows from $F_{34} = 0$ (5) that $z_1 = 5^{2(\varkappa+\delta)} z_2$, where z_2 is an integer. Substituting this in $F_{24} = 0$ (5) we get

$$F_{25} = 5^{\alpha - \varkappa + 2\alpha - 3\delta - (\varkappa+\delta)} x H_2 + y_2 K_{22} + 5^{\alpha+\delta+1} x y_2 L_{22} - 5^{\varkappa+\delta} z_2 M_{22} = 0 \quad (5),$$

which gives $y_2 = 5^{\varkappa+\delta} y_3$, where y_3 is an integer. From $F_{14} = 0$ (5) we get

$$F_{15} = 5^{2(\alpha-\varkappa)+2\alpha-3\delta-(\varkappa+\delta)} x H_1 + y_3 K_{13} + 5^{\alpha+\delta+1} x y_3 L_{13} - \\ - 5^{\varkappa+\delta} z_2 M_{12} = 0 \quad (5),$$

which gives $y_3 = 5^{\varkappa+\delta} y_4$, where y_4 is an integer. Hence system (36) may be written

$$(37) \quad \begin{cases} F_{16} = 5^{2(\alpha-\varkappa)+2\alpha-3\delta-2(\varkappa+\delta)} x H_1 + y_4 K_{14} + 5^{\alpha+\delta+1} x y_4 L_{14} - \\ \qquad - z_2 M_{12} = 0 \quad (5), \\ F_{26} = 5^{\alpha-\varkappa+2\alpha-3\delta-2(\varkappa+\delta)} x H_2 + 5^{\varkappa+\delta} y_4 K_{24} + 5^{\alpha+\delta+1+\varkappa+\delta} x y_4 L_{24} - \\ \qquad - z_2 M_{22} = 0 \quad (5), \\ F_{36} = 5^{2\alpha-3\delta-2(\varkappa+\delta)} x H_3 + 5^{2(\varkappa+\delta)} y_4 K_{34} + 5^{\alpha+\delta+1+2(\varkappa+\delta)} x y_4 L_{34} - \\ \qquad - z_2 M_{32} = 0 \quad (5). \end{cases}$$

If $2(\varkappa+\delta) < 2\alpha - 3\delta \leqq 4(\varkappa+\delta)$ we put $k = 2\alpha - 3\delta - 2(\varkappa+\delta)$ and get from (37)

$$(38) \quad \frac{\partial (F_{16}^{(k)}, F_{26}^{(k)}, F_{36}^{(k)})}{\partial (x, y_4, z_2)} \equiv -4 a^2 b^3 c^4 d f^3 g^2 5^k \equiv 0 \pmod{5^{k+1}}.$$

If $2\alpha - 3\delta > 4(\varkappa+\delta)$ the same argument may be repeated. In this way, after a finite number of operations, we can always find a k such that a congruence relation analogous to (38) holds. According to Theorem 3, the pentadic system (34) has thus only a finite number of integral solutions under the above assumptions.

Here H_j, K_j, L_j and M_j $(j=2, 3, 4)$ are functions of the same type as before. We have

$$h_{12}=2a^{\delta}b^{\delta}, \quad k_{12}=2c^{\delta}d^{\delta}, \quad m_{12}=2f^{\delta}g^{\delta},$$

$$h_{13}=2a^{\delta}b^{2\delta}, \quad k_{13}=2c^{\delta}d^{2\delta}, \quad m_{13}=2f^{\delta}g^{2\delta},$$

$$h_{14}=a\,b^{4}, \qquad k_{14}=c\,d^{4}, \qquad m_{14}=f\,g^{4}.$$

Evidently

$$H_j \equiv 0 \pmod 5, \quad M_j \equiv 0 \pmod 5. \tag{43}$$

From (42) we get

$$\begin{cases} F_{21}=5^{3(\alpha-\varkappa)}\,x\,H_2-5^{2\delta-3\varkappa}\,y\,K_2-5^{4\alpha-\delta-3\varkappa-1}\,x\,y\,L_2-z\,M_2=0 \quad (5), \\ F_{31}=5^{2(\alpha-\varkappa)}\,x\,H_3-5^{3\delta-2\varkappa}\,y\,K_3-5^{3\alpha-\delta-2\varkappa-1}\,x\,y\,L_3-z\,M_3=0 \quad (5), \\ F_{41}=5^{\alpha-\varkappa}\,x\,H_4-5^{4\delta-\varkappa}\,y\,K_4-5^{2\alpha-\delta-\varkappa-1}\,x\,y\,L_4-z\,M_4=0 \quad (5). \end{cases} \tag{44}$$

Since $2\delta-3\varkappa>3(\alpha-\varkappa)$ and $4\alpha-\delta-3\varkappa-1>3(\alpha-\varkappa)$ it follows from $F_{21}=0$ (5), according to (43), that $z=5^{3(\alpha-\varkappa)}z_1$, where z_1 is an integer. Substituting this in (44) we get

$$\begin{cases} F_{22}=x\,H_2-5^{2\delta-3\alpha}\,y\,K_2-5^{\alpha-\delta-1}\,x\,y\,L_2-z_1\,M_{21}=0 \quad (5), \\ F_{32}=x\,H_3-5^{3\delta-3\alpha}\,y\,K_3-5^{2\alpha-\delta-2}\,x\,y\,L_3-5^{\alpha-\varkappa}z_1\,M_{31}=0 \quad (5), \\ F_{42}=x\,H_4-5^{4\delta-\alpha}\,y\,K_4-5^{\alpha-\delta-1}\,x\,y\,L_4-5^{2(\alpha-\varkappa)}\,z_1\,M_{41}=0 \quad (5). \end{cases} \tag{45}$$

Since $4\delta-\alpha>2\alpha>2(\alpha-\varkappa)$ and $\alpha-\delta-1>2\alpha>2(\alpha-\varkappa)$ it follows from $F_{42}=0$ (5), according to (43), that $x=5^{2(\alpha-\varkappa)}x_1$, where x_1 is an integer. Substituting this in (45) and using operations analogous to those we have used before we get

$$\begin{cases} F_{23}=5^{2(\alpha-\varkappa)}\,x_1\,H_{21}-5^{2\delta-3\alpha}\,y\,K_2-5^{\alpha-\delta-1-2(\alpha-\varkappa)}\,x_1\,y\,L_{21}- \\ \qquad\qquad -z_1\,M_{21}=0 \quad (5), \\ F_{33}=5^{\delta-\varkappa}\,x_1\,H_{31}-5^{2\delta-3\alpha-\delta-\varkappa}\,y\,K_3-5^{\alpha-\delta-1-\alpha-\varkappa}\,x_1\,y\,L_{31}- \\ \qquad\qquad -z_1\,M_{31}=0 \quad (5), \\ F_{43}=x_1\,H_{41}-5^{2\delta-3\alpha-2(\delta-\varkappa)}\,y\,K_4-5^{\alpha-\delta-1}\,x_1\,y\,L_{41}- \\ \qquad\qquad -z_1\,M_{41}=0 \quad (5). \end{cases} \tag{46}$$

If $2\delta - 3\alpha \leqq 2(\alpha - \varkappa)$ we put $k = 2\delta - 3\alpha$ and get from (46)

$$\frac{\partial(F_{23}^{(k)}, F_{33}^{(k)}, F_{43}^{(k)})}{\partial(x_1, y, z_1)} \equiv -4\,a\,b^4\,c^3\,d^2\,f^2\,g^3\,5^k \not\equiv 0 \pmod{5^{k+1}}. \tag{47}$$

If $2\delta - 3\alpha > 2(\alpha - \varkappa)$ it follows from $F_{23} = 0$ (5) that $z_1 = 5^{2(\alpha-\varkappa)} z_2$, where z_2 is an integer. Substituting this in $F_{43} = 0$ (5) we find that $x_1 = 5^{2(\alpha-\varkappa)} x_2$, where x_2 is an integer. Hence we get from (46) the following system

$$\left\{\begin{aligned} &F_{24} = 5^{2(\alpha-\varkappa)} x_2 H_{22} + 5^{2\delta-3\alpha-2(\alpha-\varkappa)} y K_2 + 5^{\alpha+\delta+1+2(\alpha-\varkappa)} x_2 y L_{22} - \\ &\qquad - z_2 M_{22} = 0 \quad (5), \\ &F_{34} = 5^{\alpha-\varkappa} x_2 H_{32} + 5^{2\delta-3\alpha+\delta+\varkappa-2(\alpha-\varkappa)} y K_3 + 5^{\alpha+\delta+1+\alpha-\varkappa} x_2 y L_{32} - \\ &\qquad - z_2 M_{32} = 0 \quad (5), \\ &F_{44} = x_2 H_{42} + 5^{2\delta-3\alpha+2(\delta+\varkappa)-2(\alpha-\varkappa)} y K_4 + 5^{\alpha+\delta+1} x_2 y L_{42} - \\ &\qquad - z_2 M_{42} = 0 \quad (5). \end{aligned}\right. \tag{48}$$

If $2(\alpha - \varkappa) < 2\delta - 3\alpha \leqq 4(\alpha - \varkappa)$ we put $k = 2\delta - 3\alpha - 2(\alpha - \varkappa)$ and get from (48) again a congruence relation of the form (47), in fact we have only to substitute x_2 for x_1 and z_2 for z_1 in (47). If $2\delta - 3\alpha > 4(\alpha - \varkappa)$ the same argument may be repeated. In this way, after a finite number of operations, we can always find a k such that a congruence relation analogous to (47) holds. According to Theorem 3, we have thus proved that the pentadic system (42) has only a finite number of integral solutions.

31.2. Let us next suppose that $3\alpha \geqq 2\delta$. From (4) and (5) we get the pentadic system

$$\left\{\begin{aligned} &F_2 \equiv 5^{3\alpha-1} x H_2 + 5^{2\delta-1} y K_2 + 5^{\alpha-3\delta-2} x y L_2 - 5^{3\varkappa+1} z M_2 \equiv 0 \quad (5), \\ &F_3 \equiv 5^{2\alpha-1} x H_3 + 5^{3\delta-1} y K_3 + 5^{3\alpha-\delta-2} x y L_3 - 5^{2\varkappa+1} z M_3 \equiv 0 \quad (5), \\ &F_4 = 5^{\alpha+1} x H_4 + 5^{4\delta+1} y K_4 + 5^{2\alpha+\delta+2} x y L_4 - 5^{\varkappa+1} z M_4 = 0 \quad (5). \end{aligned}\right. \tag{49}$$

For $k = \alpha + 2\delta + 2\varkappa + 3$ we get from (49)

$$\frac{\partial(F_2^{(k)}, F_3^{(k)}, F_4^{(k)})}{\partial(x, y, z)} \equiv -4\,a\,b^4\,c^3\,d^2\,f^2\,g^3\,5^k \not\equiv 0 \pmod{5^{k+1}}.$$

Hence, according to Theorem 3, system (49) has only a finite number of integral solution.

32. Finally let us consider the case 3 c) in no. **24**, where

$$\delta > \alpha = \varkappa \geqq 0.$$

32.1. Let us first suppose that $2\delta > 3\alpha$. From (42) we get the system

$$\left\{\begin{aligned} F_2 &= 5^{3\alpha+1} U_2 + 5^{2\delta+1} y K_2 + 5^{4\alpha+\delta+2} x y L_2 = 0 \quad (5), \\ F_3 &= 5^{2\alpha+1} U_3 + 5^{3\delta+1} y K_3 + 5^{3\alpha+\delta+2} x y L_3 = 0 \quad (5), \\ F_4 &= 5^{\alpha+1} U_4 + 5^{4\delta+1} y K_4 + 5^{2\alpha+\delta+2} x y L_4 = 0 \quad (5). \end{aligned}\right. \tag{50}$$

Here $U_j = U_j(x, z) = \sum_{i=0}^{\infty} 5^i u_{ij}(x, z)$, where u_{ij} are polynomials with coefficients which are rational integers modulo 5. We have

$$u_{02} \equiv 2a^3 b^2 x - 2f^3 g^2 z, \quad u_{03} \equiv 2a^2 b^3 x - 2f^2 g^3 z, \quad u_{04} \equiv a b^4 x - f g^4 z.$$

From (50) we get

$$\left\{\begin{aligned} F_{21} &= U_2 + 5^{2\delta-3\alpha} y K_2 + 5^{\alpha+\delta+1} x y L_2 = 0 \quad (5), \\ F_{31} &= U_3 + 5^{3\delta-2\alpha} y K_3 + 5^{\alpha+\delta+1} x y L_3 = 0 \quad (5), \\ F_{41} &\equiv U_4 + 5^{4\delta-\alpha} y K_4 + 5^{\alpha+\delta+1} x y L_4 \equiv 0 \quad (5). \end{aligned}\right. \tag{51}$$

32.1.1. Let us suppose that $\delta < 4\alpha + 1$. Then $2\delta - 3\alpha < \alpha + \delta + 1$ and if $\Delta_1 = ag - bf \not\equiv 0 \pmod 5$ we get from (51) for $k = 2\delta - 3\alpha$

$$\frac{\partial(F_{21}^{(k)}, F_{31}^{(k)}, F_{41}^{(k)})}{\partial(x, y, z)} \equiv 4ab^3c^3d^2 f g^3 \Delta_1 5^k \not\equiv 0 \pmod{5^{k+1}}.$$

If $\Delta_1 \equiv 0 \pmod{5^\mu}$ and $\Delta_1 \not\equiv 0 \pmod{5^{\mu+1}}$, where μ is a natural number, we may write

$$a = f + a_1 5^\mu, \quad b = g + b_1 5^\mu, \tag{52}$$

where a_1 and b_1 are integers, according to Lemma 2. $\Delta_1 = \Delta_1' 5^\mu$, where $\Delta_1' = a_1 g - b_1 f \not\equiv 0 \pmod 5$. Substituting (52) in (51) we get

$$\left\{\begin{aligned} F_{21} &= (x-z) N_2 + 5^\mu T_2 + 5^{2\delta-3\alpha} y K_2 + 5^{\alpha+\delta+1} x y L_2 = 0 \quad (5), \\ F_{31} &\equiv (x-z) N_3 + 5^\mu T_3 + 5^{3\delta-2\alpha} y K_3 + 5^{\alpha+\delta+1} x y L_3 \equiv 0 \quad (5), \\ F_{41} &= (x-z) N_4 + 5^\mu T_4 + 5^{4\delta-\alpha} y K_4 + 5^{\alpha+\delta+1} x y L_4 = 0 \quad (5). \end{aligned}\right. \tag{53}$$

Here $N_j = N_j(x, z) = \sum_{i=0}^{\infty} 5^i n_{ij}(x, z)$ and $T_j = T_j(x) = \sum_{i=0}^{\infty} 5^i t_{ij}(x)$ where n_{ij} and t_{ij} are polynomials with coefficients which are rational integers modulo 5. We have

$$(54)\qquad \begin{cases} n_{02} = 2 f^3 g^2, & t_{02} = -4 f^2 g \Delta_1' x, \\ n_{03} = 2 f^2 g^3, & t_{03} = 4 f g^2 \Delta_1' x, \\ n_{04} = f g^4, & t_{04} = -4 g^3 \Delta_1' x. \end{cases}$$

Hence $N_j \equiv 0 \pmod 5$. Further we have $K_j \not\equiv 0 \pmod 5$.

If $\mu \leqq \alpha + \delta$, it follows from the relation $F_{41} = 0$ (5) that

$$x - z \equiv 0 \pmod{5^{\mu}}$$

since $4\delta - \alpha \geqq \alpha + \delta + 1 > \mu$. For $k = 2\delta - 3\alpha + \mu$ we get from (53)

$$(55)\qquad \frac{\partial(F_{21}^{(k)}, F_{31}^{(k)}, F_{41}^{(k)})}{\partial(x, y, z)} \equiv 24 c^3 d^2 f^2 g^6 \Delta_1' 5^k \not\equiv 0 \pmod{5^{k+1}}.$$

If $\mu > \alpha + \delta$ it follows from $F_{41} = 0$ (5) that $x - z \equiv 0 \pmod{5^{\alpha+\delta+1}}$. From $F_{21} = 0$ (5) we then get $5^{2\delta-3\alpha} y \equiv 0 \pmod{5^{\alpha+\delta+1}}$. Thus we may write $y = 5^{4\alpha-\delta+1} y_1$, where y_1 is an integer. Substituting this in (53) we get

$$(56)\qquad \begin{cases} F_{21} = (x - z) N_2 + 5^{\mu} T_2 + 5^{\alpha+\delta+1} y_1 K_{21} + 5^{5\alpha+2} x y_1 L_{21} = 0 \quad (5), \\ F_{31} = (x - z) N_3 + 5^{\mu} T_3 + 5^{2(\alpha+\delta)+1} y_1 K_{31} + 5^{5\alpha+2} x y_1 L_{31} = 0 \quad (5), \\ F_{41} = (x - z) N_4 + 5^{\mu} T_4 + 5^{3(\alpha+\delta)+1} y_1 K_{41} + 5^{5\alpha+2} x y_1 L_{41} = 0 \quad (5). \end{cases}$$

If $\mu = \alpha + \delta + 1$ we put $k = \alpha + \delta + 1 + \mu = 2\mu$ and get from (56) a relation analogous to (55). If $\mu > \alpha + \delta + 1$ it follows from $F_{41} = 0$ (5) that $x - z \equiv 0 \pmod{5^{\alpha+\delta+2}}$ since $5\alpha + 2 \geqq \alpha + \delta + 2$. By $F_{21} = 0$ (5) we then get $y_1 = 5 y_2$, where y_2 is an integer. Substituting this in (56) we again get, when $\mu = \alpha + \delta + 2$ for $k = 2\mu$, a congruence relation analogous to (55). If $\mu > \alpha + \delta + 2$ this argument may be repeated. In this way, after a finite number of operations, we can always find a k such that a congruence relation analogous to (55) holds. We have thus treated the case $\delta < 4\alpha + 1$ completely.

32.1.2. Now let us suppose that $\delta \geqq 4\alpha+1$. In this case we consider the following system[1], deduced from (4) and (5)

$$(57)\qquad \begin{cases} F_{11} = U_1 - 5^{\delta-4\alpha}\, y\, K_1 - 5^{\alpha-\delta-2}\, x\, y\, L_1 = 0 \quad (5), \\ F_{21} = U_2 + 5^{2\delta-3\alpha}\, y\, K_2 + 5^{\alpha+\delta+1}\, x\, y\, L_2 = 0 \quad (5), \\ F_{31} = U_3 - 5^{3\delta-2\alpha}\, y\, K_3 - 5^{\alpha+\delta-1}\, x\, y\, L_3 = 0 \quad (5), \end{cases}$$

where U_1, K_1 and L_1 are functions of the same type as before. We have $u_{01} = a^4 b\, x - f^4 g\, z$ and $k_{01} = c^4 d$. If $\Delta_1 = a\, g - b\, f \not\equiv 0 \pmod 5$ we get from (57) for $k = \delta - 4\alpha$.

$$\frac{\partial(F_{11}^{(k)}, F_{21}^{(k)}, F_{31}^{(k)})}{\partial(x, y, z)} \equiv 4\, a^2 b^2 c^4 d\, f^2 g^2\, \Delta_1\, 5^k \not\equiv 0 \pmod{5^{k+1}}.$$

If $\Delta_1 \equiv 0 \pmod{5^\mu}$ and $\Delta_1 \not\equiv 0 \pmod{5^{\mu+1}}$, where μ is a natural number, then (52) holds. Using the same notations as before we thus get from (57)

$$(58)\qquad \begin{cases} F_{11} = (x-z)\, N_1 + 5^\mu\, T_1 + 5^{\delta-4\alpha}\, y\, K_1 + 5^{\alpha+\delta+2}\, x\, y\, L_1 = 0 \quad (5), \\ F_{21} = (x-z)\, N_2 + 5^\mu\, T_2 + 5^{2\delta-3\alpha}\, y\, K_2 + 5^{\alpha+\delta+1}\, x\, y\, L_2 = 0 \quad (5), \\ F_{31} = (x-z)\, N_3 + 5^\mu\, T_3 + 5^{3\delta-2\alpha}\, y\, K_3 + 5^{\alpha+\delta+1}\, x\, y\, L_3 = 0 \quad (5). \end{cases}$$

Here $n_{01} = f^4 g$ and $t_{01} = 4\, f^3\, \Delta_1' x$ holds as well as (54).

If $\mu \leqq \alpha+\delta$ it follows from $F_{31} = 0$ (5) that $x - z \equiv 0 \pmod{5^\mu}$ since $3\delta - 2\alpha > 2\delta - 3\alpha \geqq \alpha+\delta+1$. For $k = \delta - 4\alpha + \mu$ we get from (58)

$$(59)\qquad \frac{\partial(F_{11}^{(k)}, F_{21}^{(k)}, F_{31}^{(k)})}{\partial(x, y, z)} \equiv -16\, c^4 d\, f^4 g^4\, \Delta_1'\, 5^k \not\equiv 0 \pmod{5^{k+1}}.$$

If $\mu > \alpha+\delta$ it follows from $F_{31} = 0$ (5) that $x - z \equiv 0 \pmod{5^{\alpha+\delta+1}}$. From $F_{11} = 0$ (5) we thus get $y = 5^{5\alpha+1}\, y_1$, where y_1 is an integer. Substituting this in (58) we get the system

[1] We could also have used the functions F_1, F_2 and F_3 in no. **31** but it had then been necessary to distinguish between the cases $\delta < 4\alpha+1$ and $\delta \geqq 4\alpha+1$ already in that no.

$$(60)\qquad \begin{cases} F_{11} = (x-z)\,N_1 + 5^{\mu}\,T_1 + 5^{\alpha+\delta+1}\,y_1\,K_{11} + 5^{6\alpha+\delta+3}\,x\,y_1\,L_{11} = 0 \quad (5), \\ F_{21} = (x-z)\,N_2 + 5^{\mu}\,T_2 + 5^{2(\alpha+\delta)+1}\,y_1\,K_{21} + \\ \qquad\qquad + 5^{6\alpha+\delta+2}\,x\,y_1\,L_{21} = 0 \quad (5), \\ F_{31} = (x-z)\,N_3 + 5^{\mu}\,T_3 + 5^{3(\alpha+\delta)+1}\,y_1\,K_{31} + \\ \qquad\qquad + 5^{6\alpha+\delta+2}\,x\,y_1\,L_{31} = 0 \quad (5). \end{cases}$$

If $\mu = \alpha+\delta+1$ we put $k = \alpha+\delta+1 \div \mu = 2\mu$ and get from (60) a congruence relation analogous to (59). If $\mu > \alpha+\delta+1$, exactly the same reasoning as for system (56) may be applied. Hence we can always find a k such that a congruence relation analogous to (59) holds. We have thus treated the case $\delta > 4\alpha+1$ completely.

32.2. Let us next suppose that $3\alpha \geqq 2\delta$. From (49) we get the system

$$(61)\qquad \begin{cases} F_2 = 5^{3\alpha+1}\,U_2 + 5^{2\delta+1}\,y\,K_2 + 5^{\alpha+3\delta+2}\,x\,y\,L_2 = 0 \quad (5), \\ F_3 = 5^{2\alpha+1}\,U_3 + 5^{3\delta+1}\,y\,K_3 + 5^{3\alpha+\delta+2}\,x\,y\,L_3 = 0 \quad (5), \\ F_4 = 5^{\alpha+1}\,U_4 + 5^{4\delta+1}\,y\,K_4 + 5^{2\alpha+\delta+2}\,x\,y\,L_4 = 0 \quad (5), \end{cases}$$

where the notations are the same as before. From (61) we get

$$\begin{cases} F_{21} = 5^{3\alpha-2\delta}\,U_2 + y\,K_2 + 5^{\alpha+\delta+1}\,x\,y\,L_2 = 0 \quad (5), \\ F_{31} = U_3 + 5^{3\delta-2\alpha}\,y\,K_3 + 5^{\alpha+\delta+1}\,x\,y\,L_3 = 0 \quad (5), \\ F_{41} = U_4 + 5^{4\delta-\alpha}\,y\,K_4 + 5^{\alpha+\delta+1}\,x\,y\,L_4 = 0 \quad (5). \end{cases}$$

Since $3\alpha - 2\delta < \alpha+\delta+1$ it follows from $F_{21} = 0$ (5) that $y = 5^{3\alpha-2\delta}\,y_1$, where y_1 is an integer. Hence we get

$$(62)\qquad \begin{cases} F_{22} = U_2 + y_1\,K_{21} + 5^{\alpha+\delta+1}\,x\,y_1\,L_{21} = 0 \quad (5), \\ F_{32} = U_3 + 5^{\alpha+\delta}\,y_1\,K_{31} + 5^{4\alpha-\delta+1}\,x\,y_1\,L_{31} = 0 \quad (5), \\ F_{42} = U_4 + 5^{2(\alpha-\delta)}\,y_1\,K_{41} + 5^{4\alpha-\delta+1}\,x\,y_1\,L_{41} = 0 \quad (5). \end{cases}$$

If $\Delta_1 = a\,g - b\,f \not\equiv 0 \pmod 5$ we get from (62)

$$\frac{\partial\,(F_{22}^{(0)},\,F_{32}^{(0)},\,F_{42}^{(0)})}{\partial\,(x,\,y_1,\,z)} = 4\,a\,b^3\,c^3\,d^2\,f\,g^3\,\Delta_1 \not\equiv 0 \pmod 5.$$

If $\Delta_1 \equiv 0 \pmod{5^\mu}$ and $\Delta_1 \not\equiv 0 \pmod{5^{\mu+1}}$, where μ is a natural number, then (52) holds. Using the same notations as before we thus get from (62)

$$\left|\begin{array}{l} F_{22} = (x-z)\,N_2 + 5^\mu\,T_2 + y_1\,K_{21} + 5^{\alpha+\delta+1}\,x\,y_1\,L_{21} = 0 \quad (5), \\ F_{32} = (x-z)\,N_3 + 5^\mu\,T_3 + 5^{\alpha+\delta}\,y_1\,K_{31} + 5^{4\alpha-\delta+1}\,x\,y_1\,L_{31} = 0 \quad (5), \\ F_{42} = (x-z)\,N_4 + 5^\mu\,T_4 + 5^{2(\alpha-\delta)}\,y_1\,K_{41} + 5^{4\alpha-\delta+1}\,x\,y_1\,L_{41} = 0 \quad (5). \end{array}\right.$$

If $\mu \leqq \alpha+\delta$ it follows from $F_{42}=0$ (5) that $x-z \equiv 0 \pmod{5^\mu}$ since $4\alpha-\delta+1 > \alpha+\delta \geqq \mu$. For $k=\mu$ we get, according to (54),

$$(63) \qquad \frac{\partial(F_{22}^{(k)}, F_{32}^{(k)}, F_{42}^{(k)})}{\partial(x, y_1, z)} \equiv 24\,c^3\,d^2\,f^2\,g^6\,\Delta_1'\,5^k \not\equiv 0 \pmod{5^{k+1}}.$$

If $\mu > \alpha+\delta$ it follows from $F_{42}=0$ (5) that $x-z \equiv 0 \pmod{5^{\alpha+\delta+1}}$. Thus we get from $F_{22}=0$ (5) that $y_1 = 5^{\alpha+\delta+1}\,y_2$, where y_2 is an integer. Hence we get the following system

$$(64) \qquad \left|\begin{array}{l} F_{22} = (x-z)\,N_2 + 5^\mu\,T_2 + 5^{\alpha+\delta+1}\,y_2\,K_{22} + 5^{2(\alpha+\delta+1)}\,x\,y_2\,L_{22} = 0 \quad (5), \\ F_{32} = (x-z)\,N_3 + 5^\mu\,T_3 + 5^{2(\alpha+\delta)+1}\,y_2\,K_{32} + 5^{5\alpha+2}\,x\,y_2\,L_{32} = 0 \quad (5), \\ F_{42} = (x-z)\,N_4 + 5^\mu\,T_4 + 5^{3(\alpha+\delta)+1}\,y_2\,K_{42} + 5^{5\alpha+2}\,x\,y_2\,L_{42} = 0 \quad (5). \end{array}\right.$$

If $\alpha+\delta < \mu \leqq 2(\alpha+\delta)$ it follows from $F_{42}=0$ (5) that $x-z \equiv 0 \pmod{5^\mu}$ since $3(\alpha+\delta)+1 > 5\alpha+2 \geqq 2(\alpha+\delta+1) > \mu$. In this case we put $k=\alpha+\delta+1+\mu$ and get from (64) a congruence relation analogous to (63). If $\mu > 2(\alpha+\delta)$ the same reasoning may be repeated. In this way, after a finite number of operations, we can always find a k such that a congruence relation analogous to (63) holds. By Theorem 3 system (61) has thus only a finite number of integral solutions.

It follows from this paragraph that the equation

$$u^5 + M\,v^5 = 1$$

cannot have three non-trivial integral solutions u_1; v_1, u_2; v_2 and u_3; v_3 such that $v_1\,u_2\,u_3 \equiv 0 \pmod 5$ and $u_1\,v_2\,v_3 \not\equiv 0 \pmod 5$.

§ 6. $u^5 + Mv^5 = 1$ cannot have three non-trivial integral solutions $u_1; v_1, u_2; v_2$ and $u_3; v_3$ such that $v_1 v_2 u_3 \equiv 0 \pmod 5$ and $u_1 u_2 v_3 \not\equiv 0 \pmod 5$.

33. Let us consider the case in which the coefficient of θ in two of the numbers ε_1, ε_2 and ε_3 is divisible by 5, while the coefficient of θ in the third number is not divisible by 5. Clearly it is sufficient to treat the case in which

$$\varepsilon_1 = 5^\alpha a + b\theta, \quad \varepsilon_2 = c + 5^\delta d\theta, \quad \varepsilon_3 = f + 5^\lambda g\theta, \tag{1}$$

where δ and λ are $\geqq 0$ and $\alpha > 0$, i.e. the case 4° in no. **13.**

We must examine the following cases

1) $\delta \geqq \lambda \geqq \alpha > 0$, 2) $\delta \geqq \alpha \geqq \lambda \geqq 0$, 3) $\alpha \geqq \delta \geqq \lambda \geqq 0$.

By (1) we get

$$\varepsilon_1^5 = 1 + 5^{\alpha+1}\xi, \quad \varepsilon_2^5 = 1 + 5^{\delta+1}\eta, \quad \varepsilon_3^5 = 1 + 5^{\lambda+1}\zeta, \tag{2}$$

where

$$\begin{cases} \xi = 5^{3\alpha} a^4 b\theta + 2\cdot 5^{2\alpha} a^3 b^2 \theta^2 + 2\cdot 5^\alpha a^2 b^3 \theta^3 + a b^4 \theta^4, \\ \eta = c^4 d\theta + 2\cdot 5^\delta c^3 d^2 \theta^2 + 2\cdot 5^{2\delta} c^2 d^3 \theta^3 + 5^{3\delta} c d^4 \theta^4, \\ \zeta = f^4 g\theta + 2\cdot 5^\lambda f^3 g^2 \theta^2 + 2\cdot 5^{2\lambda} f^2 g^3 \theta^3 + 5^{3\lambda} f g^4 \theta^4. \end{cases} \tag{3}$$

If we compare these expressions, which are fundamental for the operations, with the corresponding expressions in no. **24**, we find that the above-mentioned formulas are symmetrical in relation to those in no. **24.** Since the proofs in this case can be carried out in a symmetrical way to the proofs in § 5, it is not necessary to work them out here. Nevertheless, it is evident that the equation $u^5 + Mv^5 = 1$ cannot have three non-trivial integral solutions u_1; v_1, u_2; v_2 and u_3; v_3 such that

$$v_1 v_2 u_3 \equiv 0 \pmod 5 \quad \text{and} \quad u_1 u_2 v_3 \not\equiv 0 \pmod 5.$$

§ 7. Summary.

34. It is shown in the preceding paragraphs that the following theorem holds:

Theorem 5. *If M is any rational integer, the Diophantine equation*

$$u^5 + M v^5 = 1$$

has at most two solutions in rational integers u and v with $v \neq 0$.

Combining this result with that of SIEGEL (Cp. Introduction, p. 5) concerning the equation $A u^n - B v^n = C$, we obtain the following

Theorem 6. *Let M be any rational integer. If $|M| \geqq 2060$, the Diophantine equation*

$$u^5 + M v^5 = 1$$

has at most one solution in rational integers u and v with $v \neq 0$. In the case of $|M| < 2060$ it has at most two solutions in rational integers u and v with $v \neq 0$.

However, it should be pointed out that this theorem which is to some extent stronger than Theorem 5 has not been proved hitherto by the exclusive use of methods of the theory of numbers.

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